Induced Arithmetic Removal Lemmas

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GRAPH REMOVAL LEMMAS

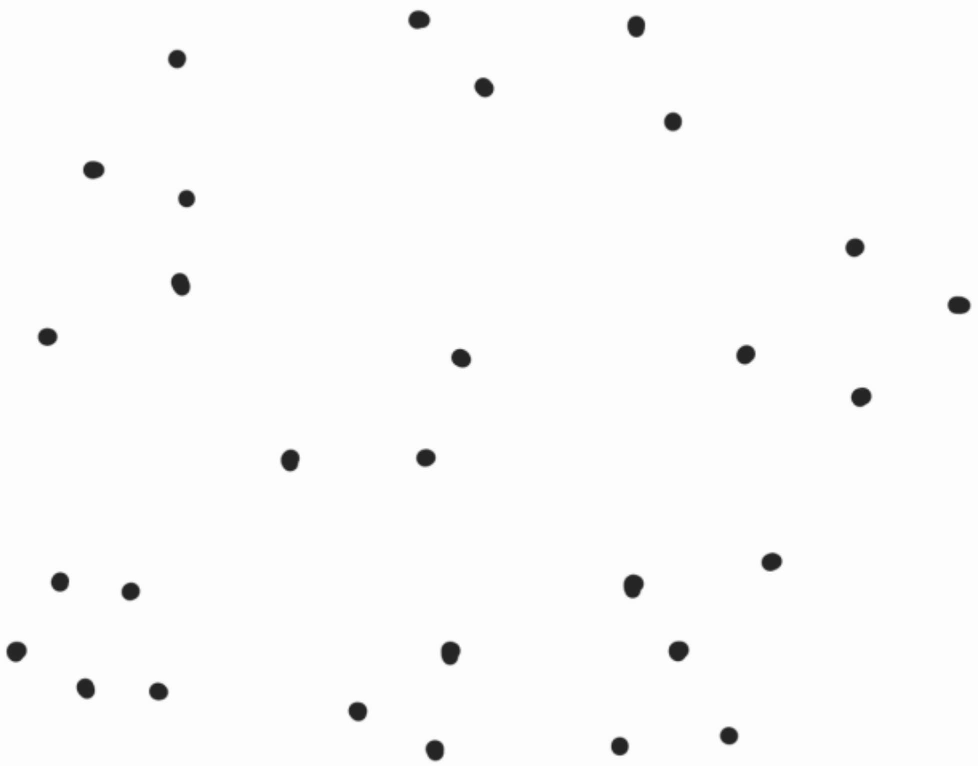
Triangle Removal (Rusza, Szemerédi 1976)

> For all $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if a graph G has $\leq \delta n^3$ triangles, G can be made Δ-free by removing $\leq \varepsilon n^2$ edges.

GRAPH REMOVAL LEMMAS

Szemerédi Regularity Lemma

(informal)



Vertex partition s.t. for almost all pairs of classes, the edge distribution ≈ uniform

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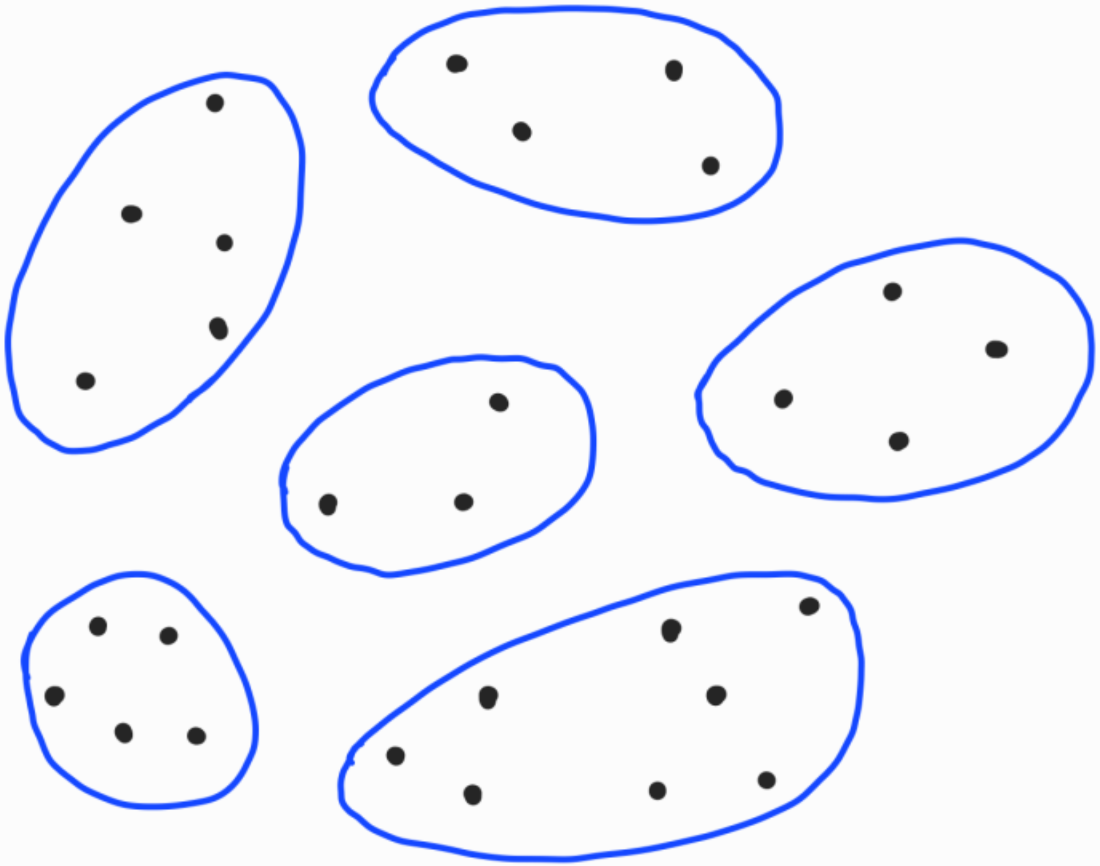

Vertex partition s.t. for almost all pairs of classes, the edge distribution $\approx$ uniform

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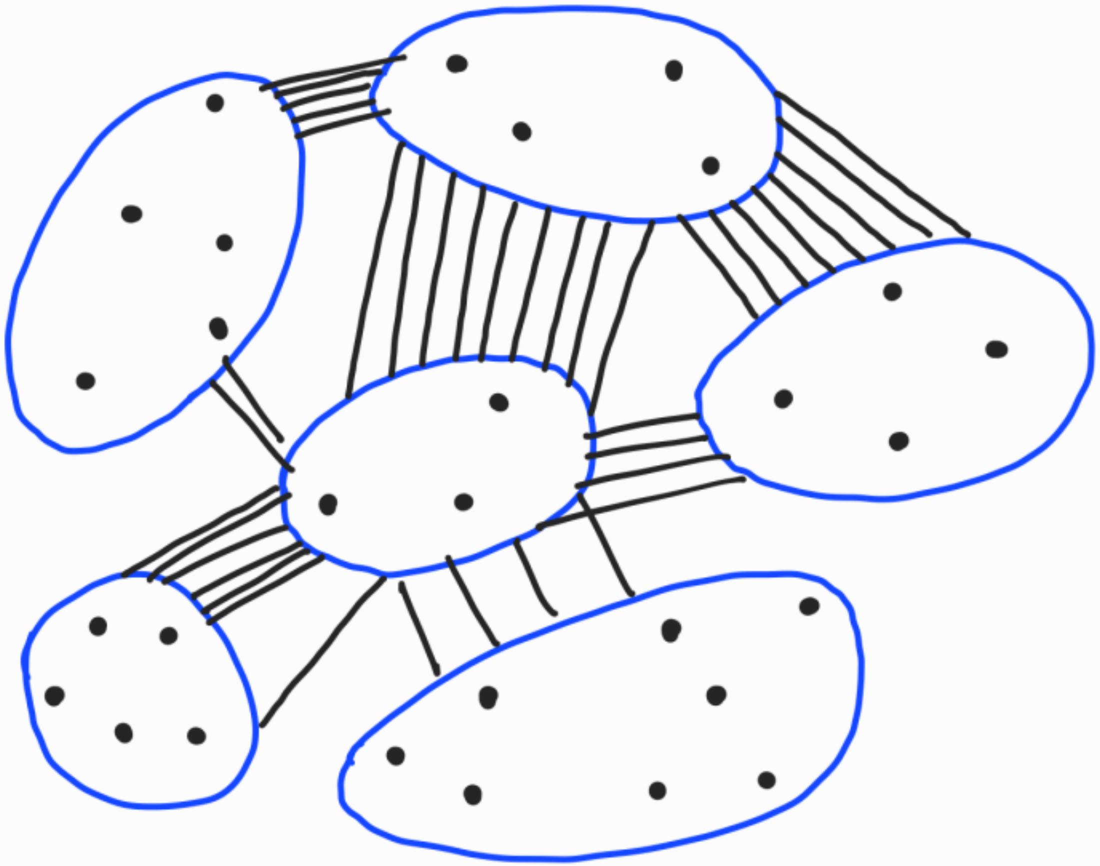

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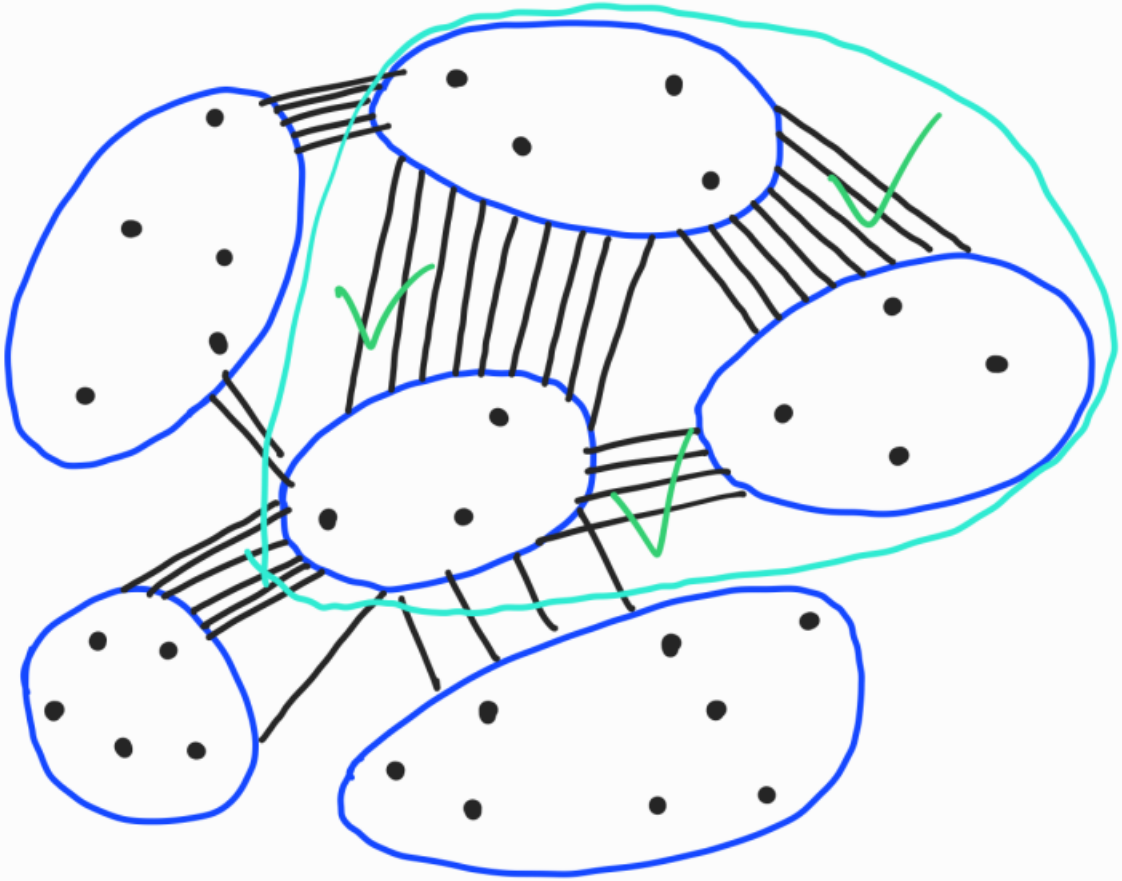

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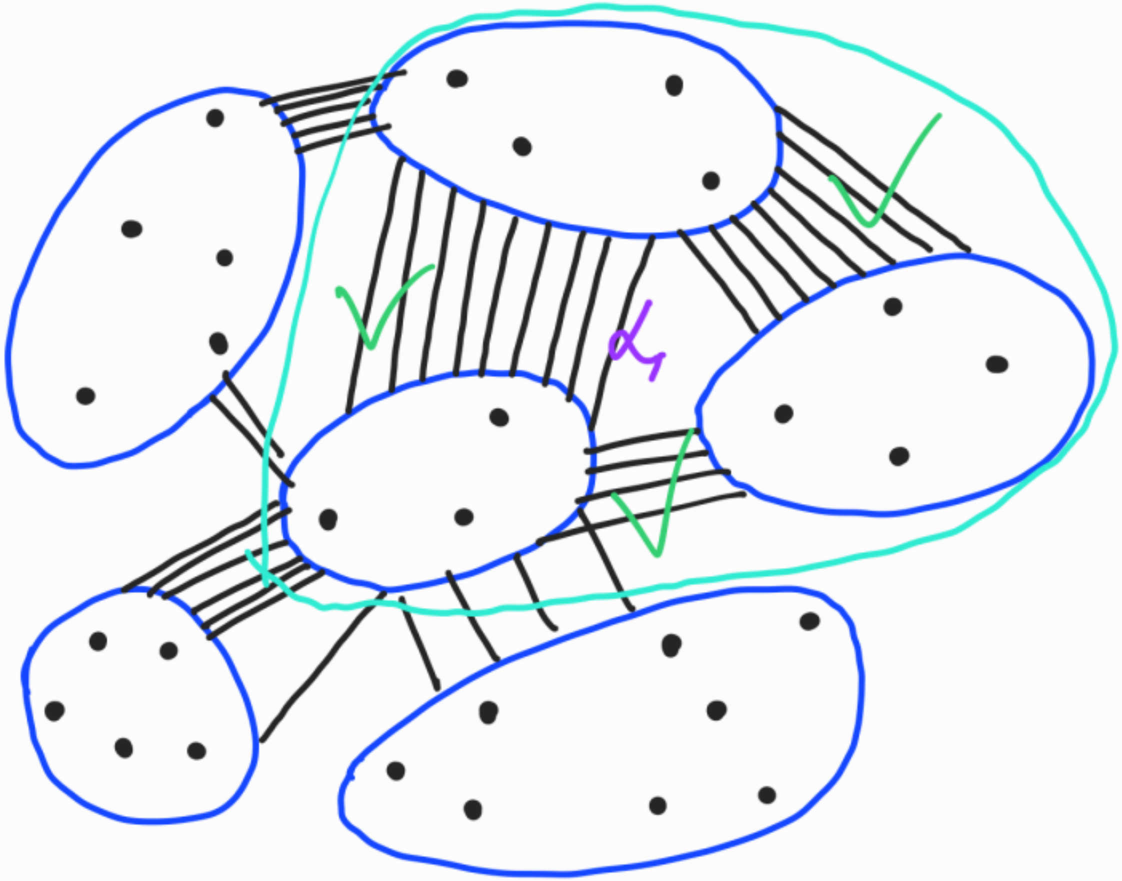

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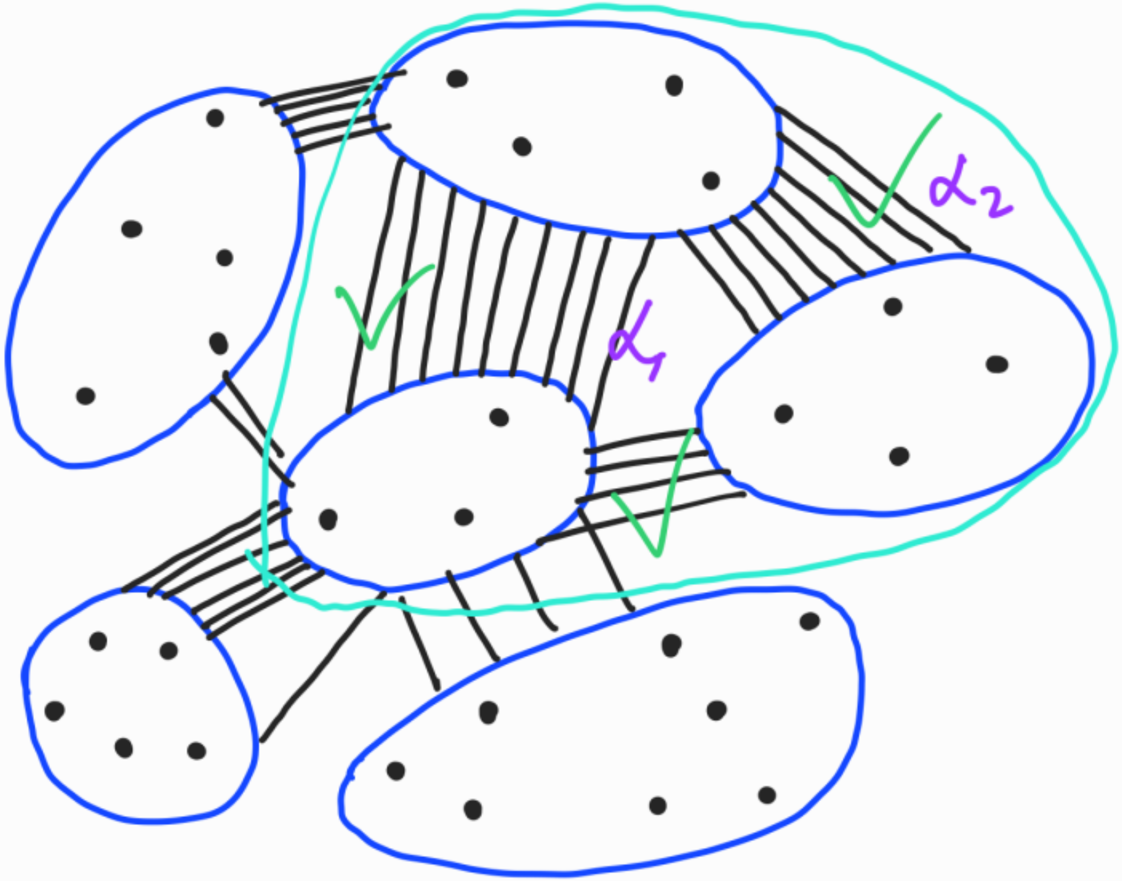

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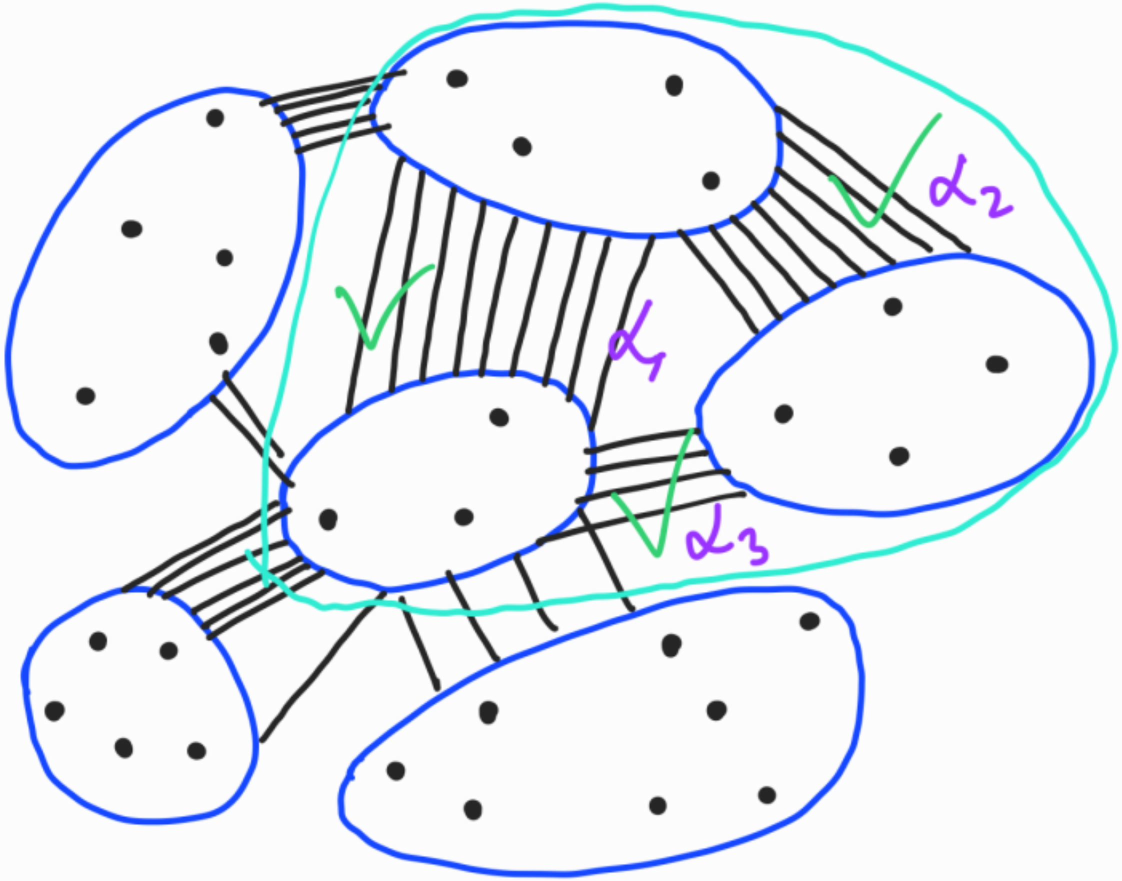

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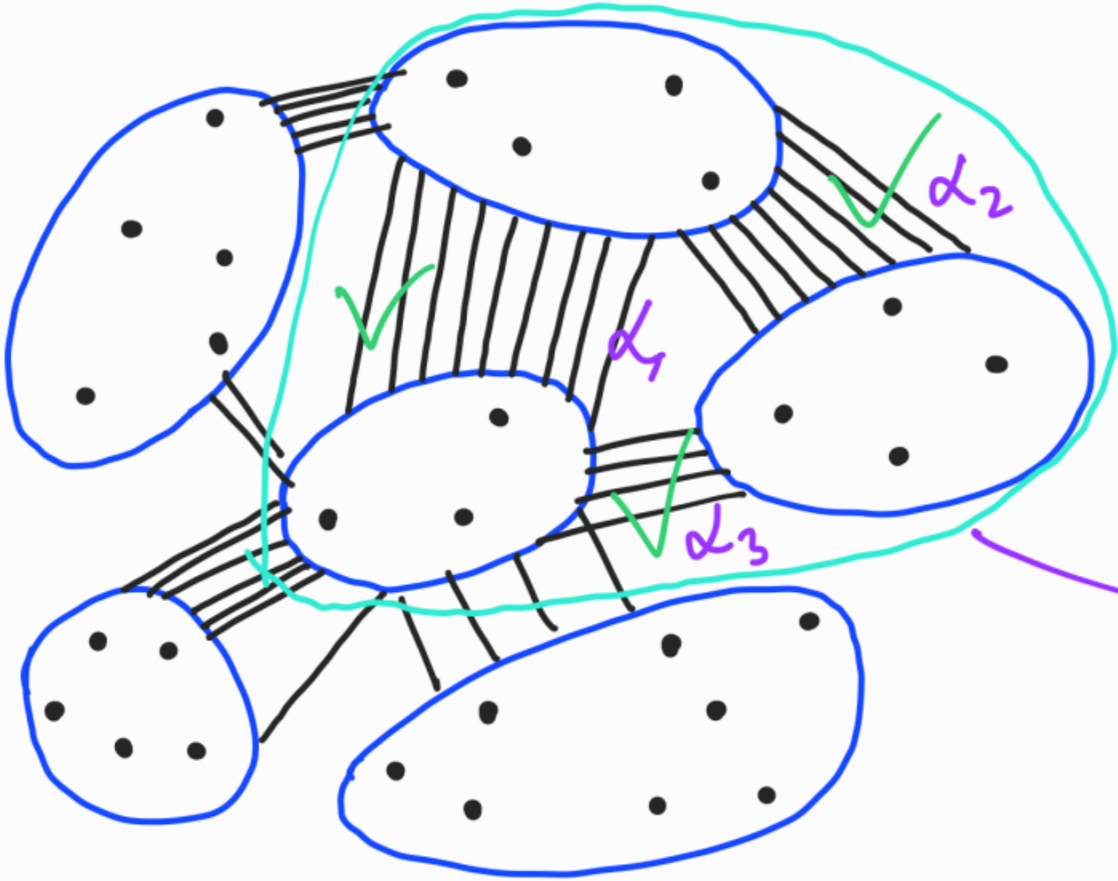

Vertex partition s.t. for almost all pairs of classes, the edge distribution $\approx$ uniform

$\Rightarrow \approx \alpha_1 \alpha_2 \alpha_3 m^3$ triangles

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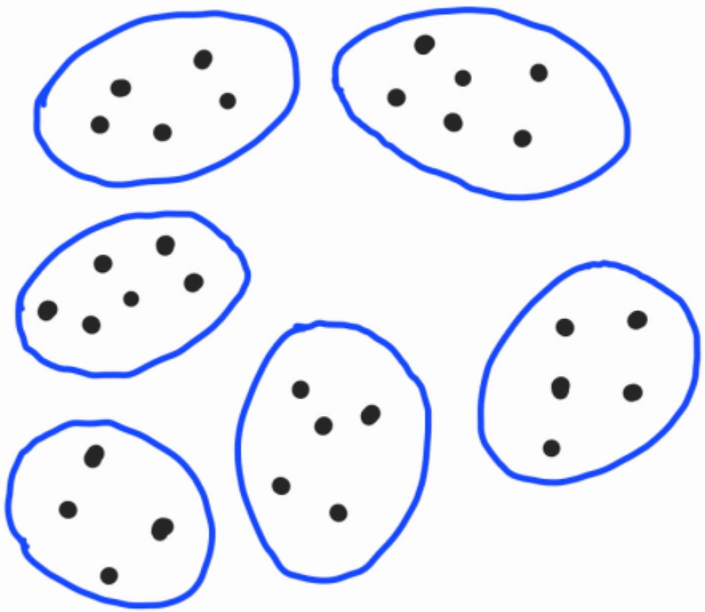

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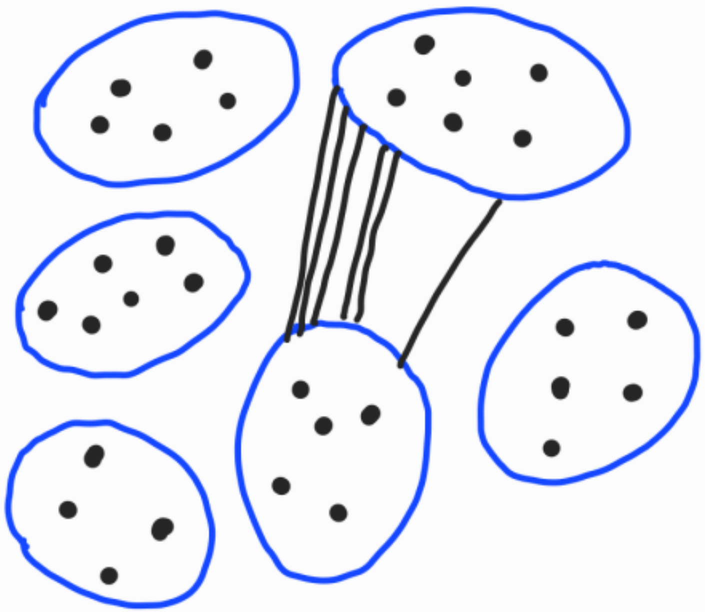
bad pairs



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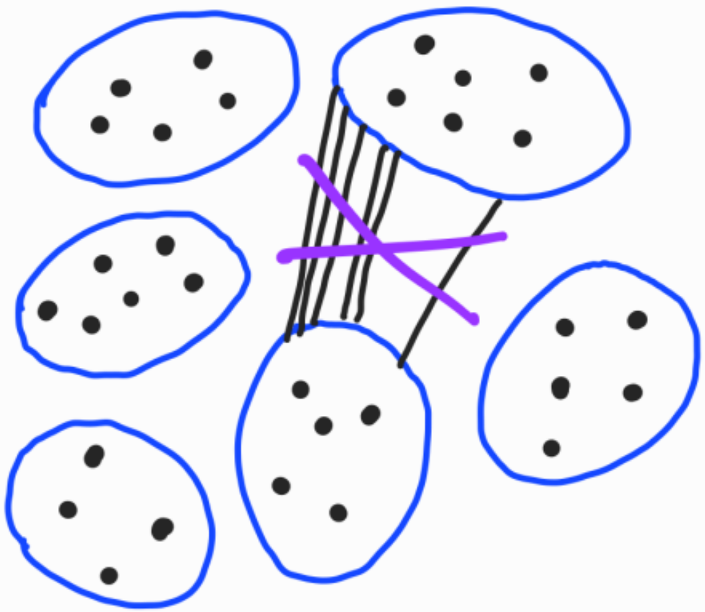

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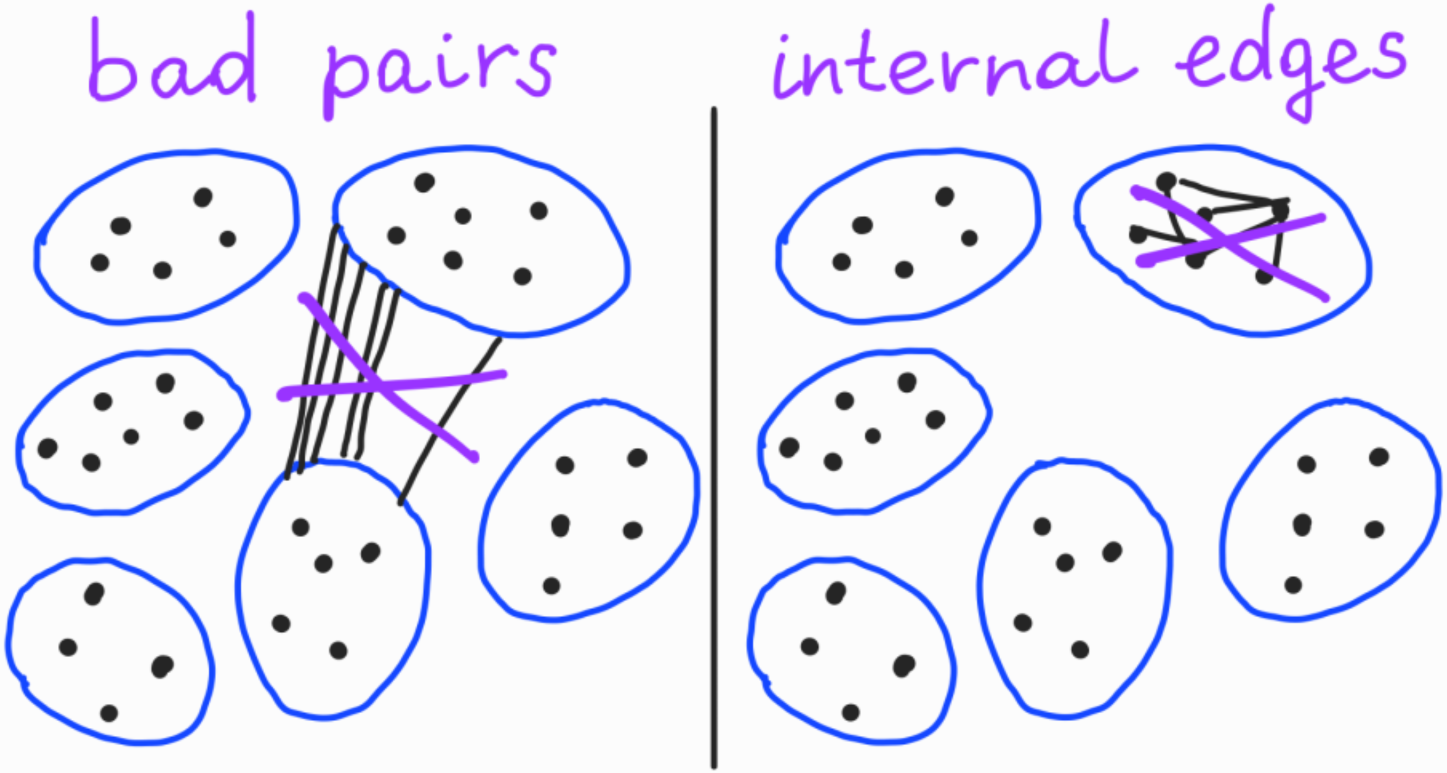

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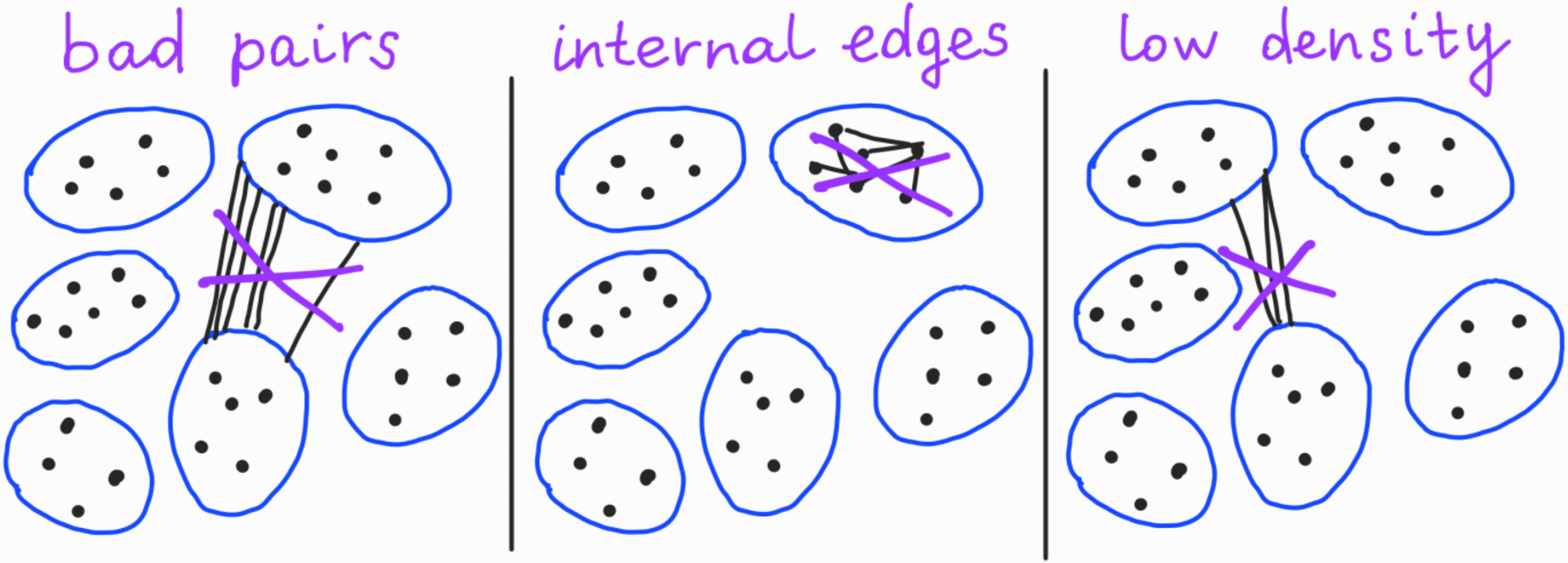

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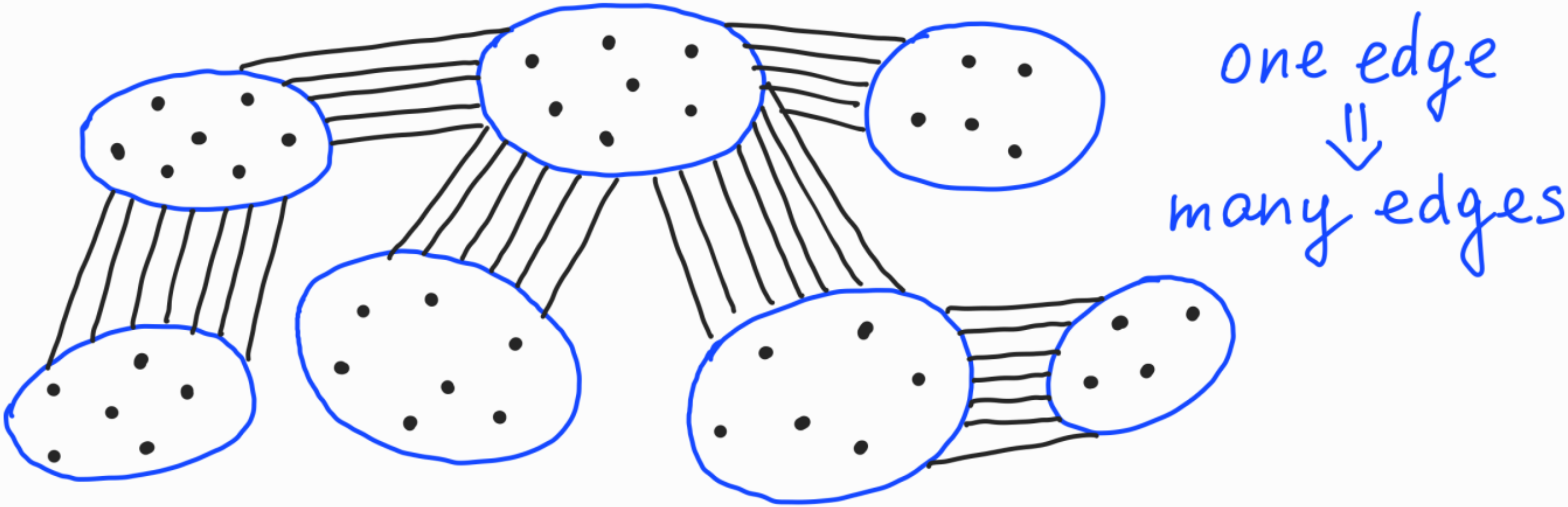

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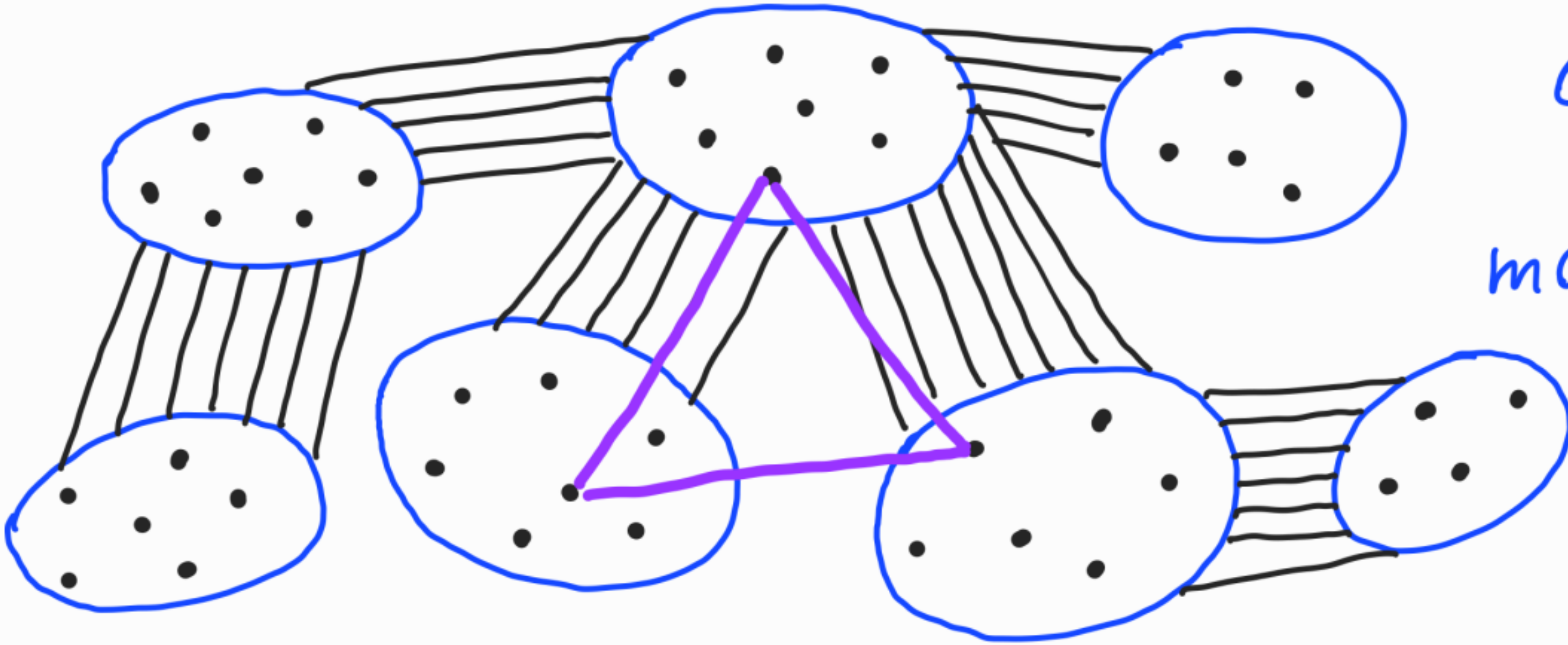

one edge

$\Downarrow$

many edges

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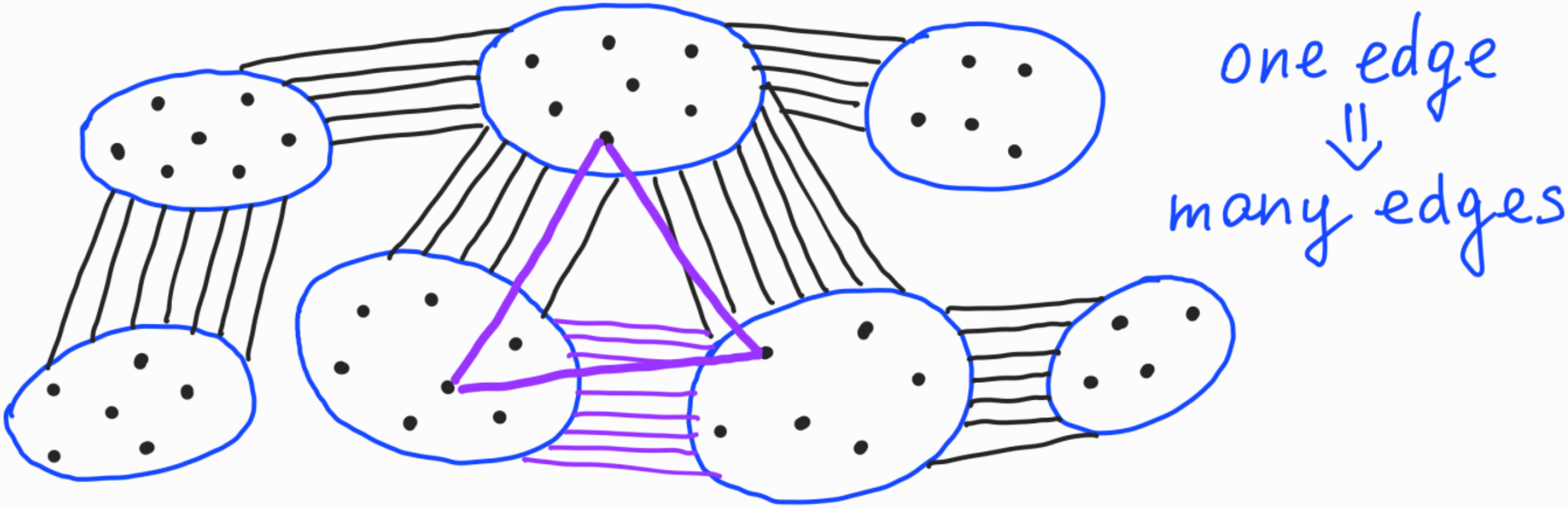

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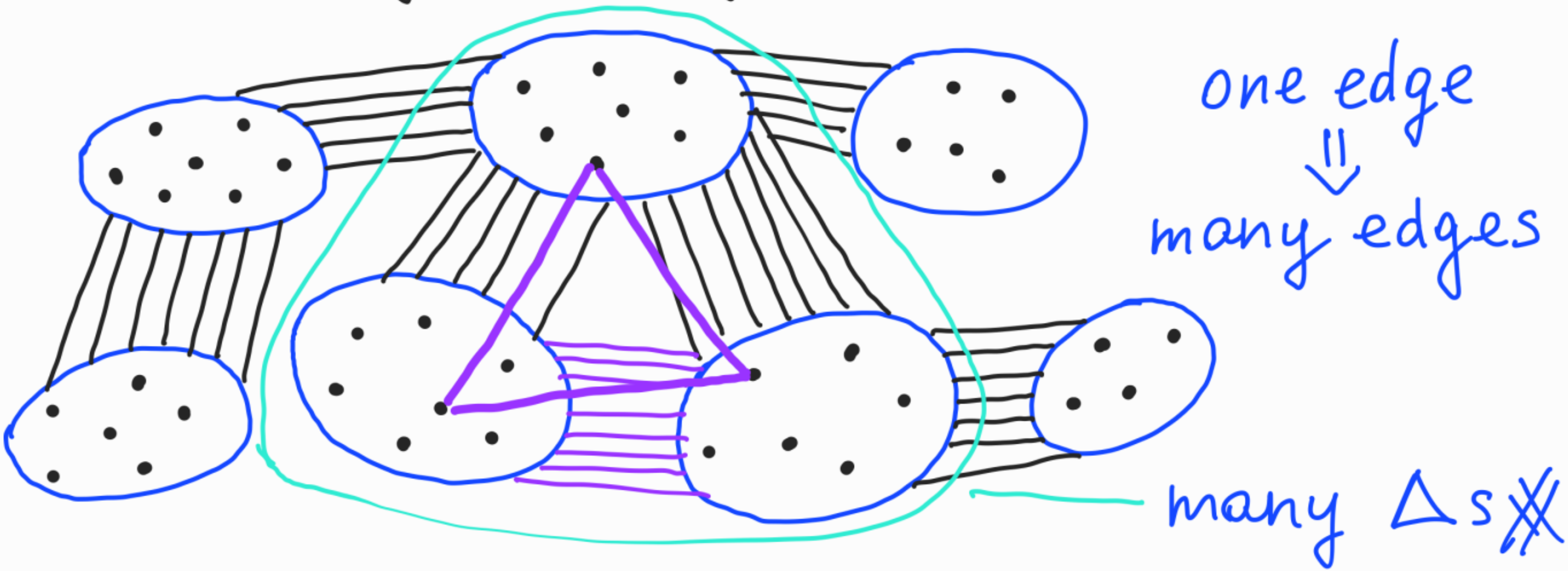

GRAPH REMOVAL LEMMAS

<u>Subgraph Removal</u> (Rusza, Szemerédi 1976)

For all $\varepsilon > 0$ and graphs H, $\exists\ \delta > 0$ s.t. if a graph G has $\leq \delta n^{|H|}$ copies of H, G can be made H-free by removing $\leq \varepsilon n^2$ edges.

<u>Same proof</u>

GRAPH REMOVAL LEMMAS

Induced Subgraph Removal

(Alon, Fischer, Krivelevich, Szegedy 2000)

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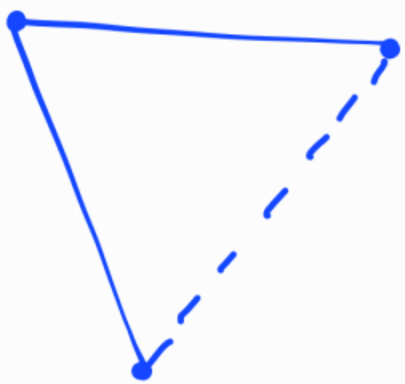
$H =$ 

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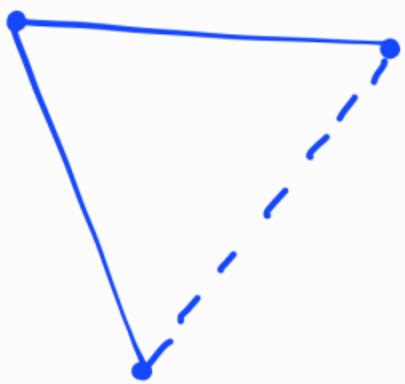
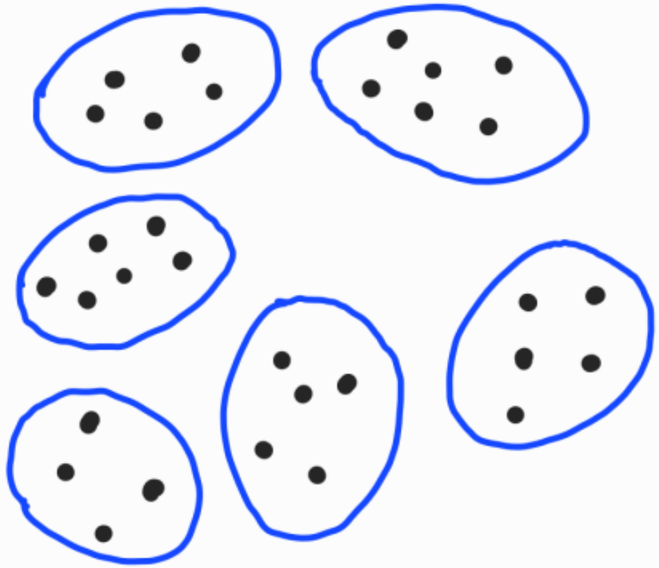
$H =$ 



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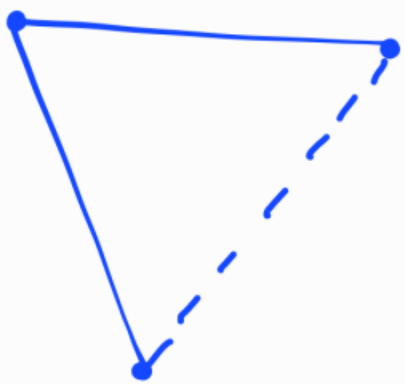
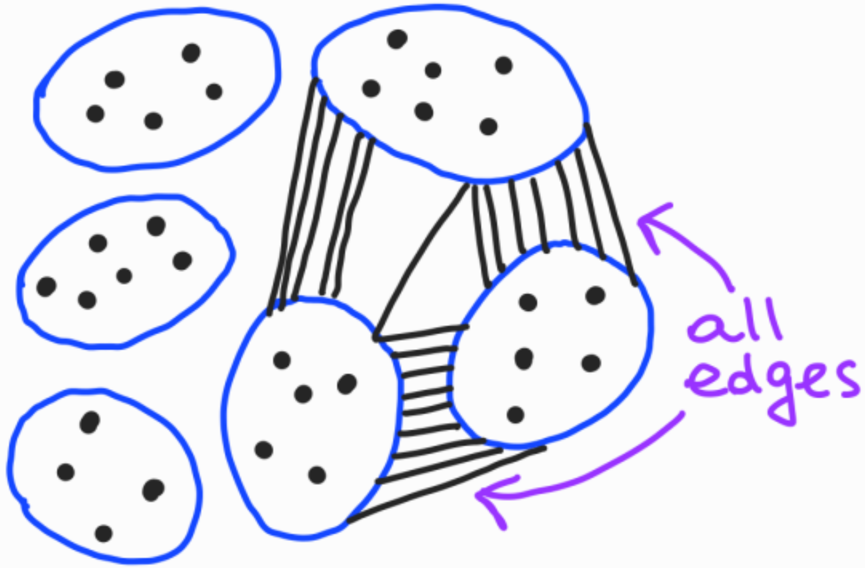
$H =$ 



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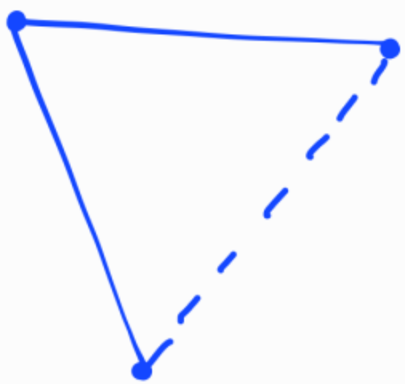
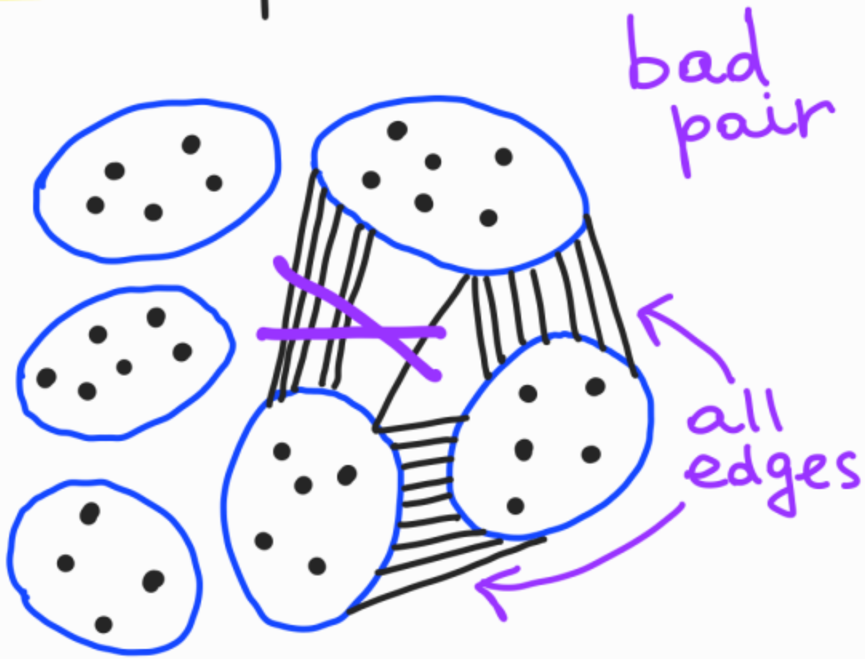
$H =$ 



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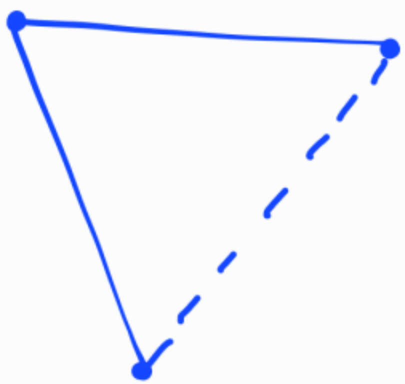
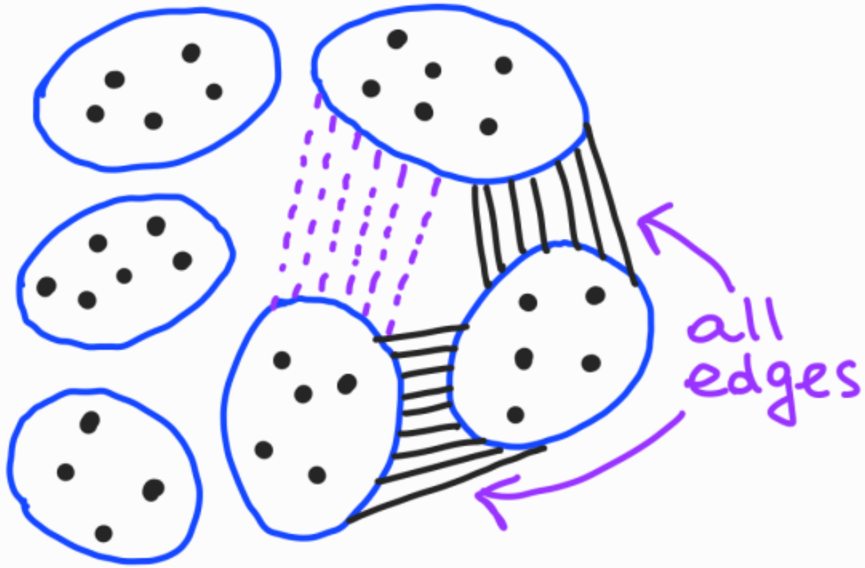
$H =$ 



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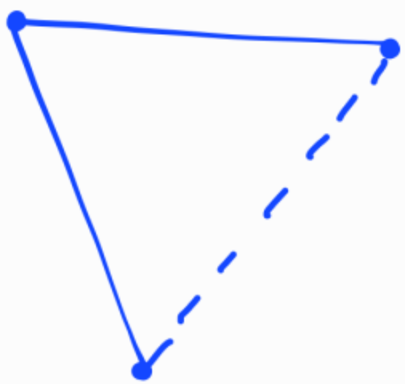
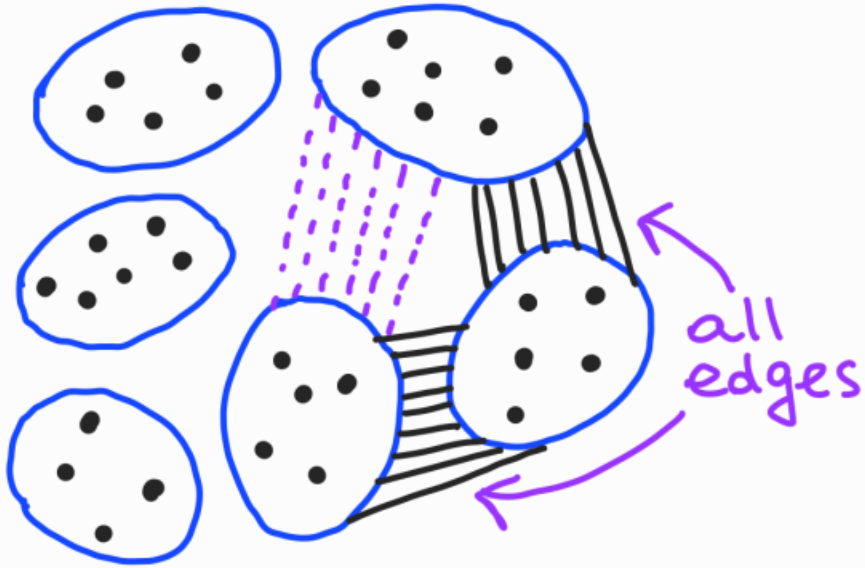
$H =$ 

Lots of new instances!



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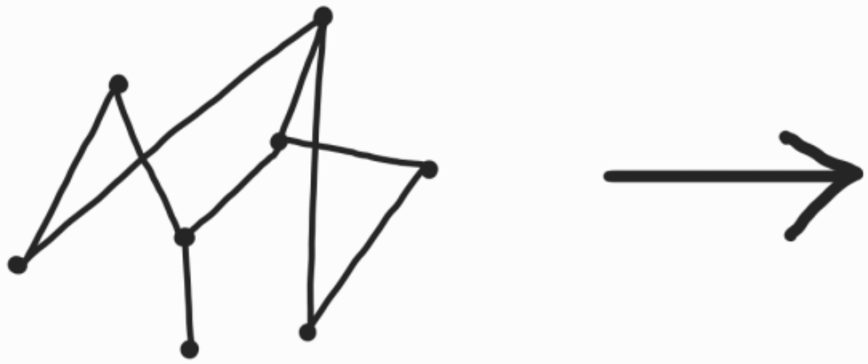
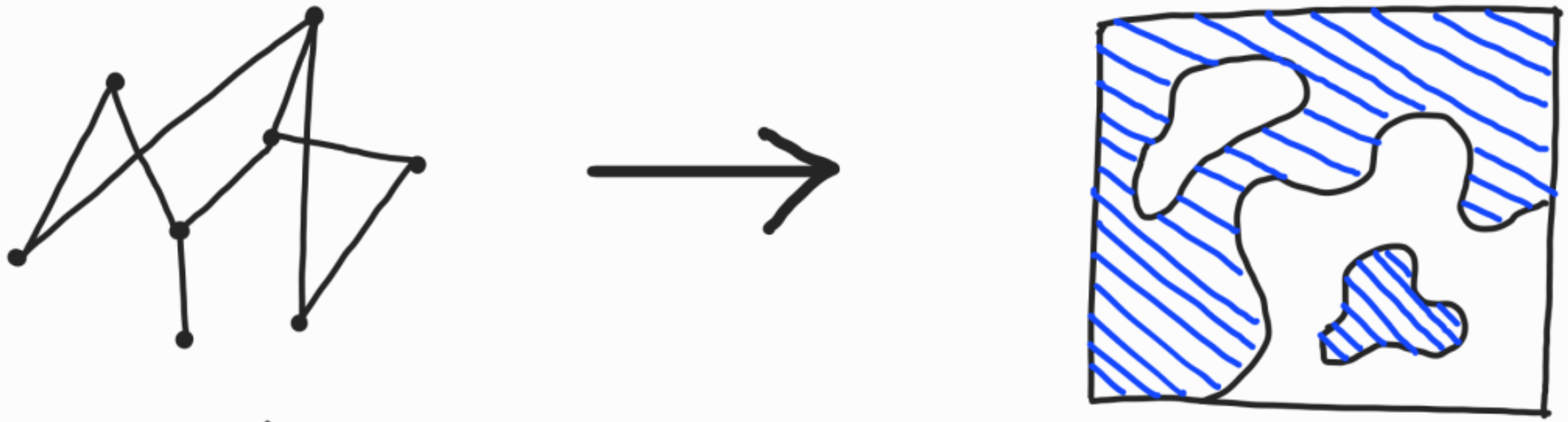
Tool: Strong Regularity Lemma

ARITHMETIC SETTING



graph

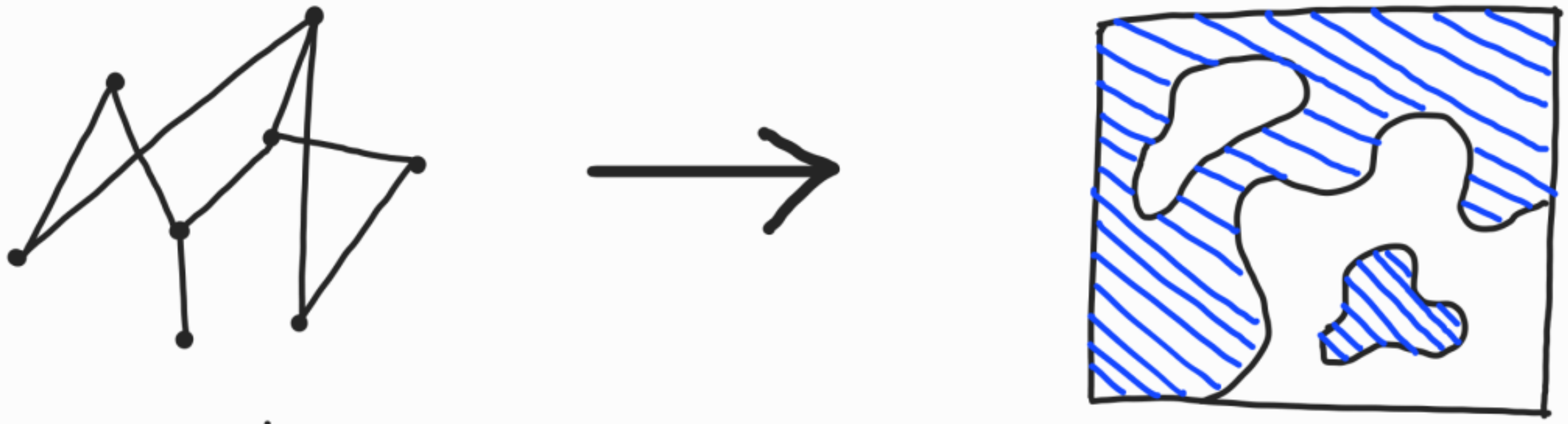
ARITHMETIC SETTING



graph

set in a group

ARITHMETIC SETTING



graph

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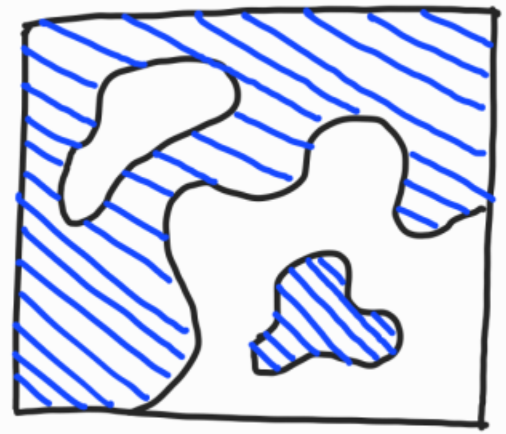
subgraphs

ARITHMETIC SETTING

graph

$\longrightarrow$



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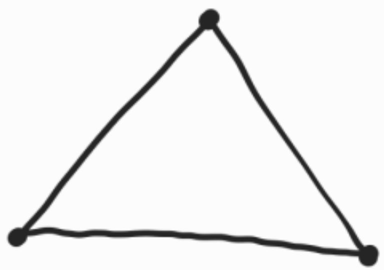

subgraphs

$\longrightarrow$

$$x - 2y + z = 0$$

solutions to linear equations

ARITHMETIC SETTING

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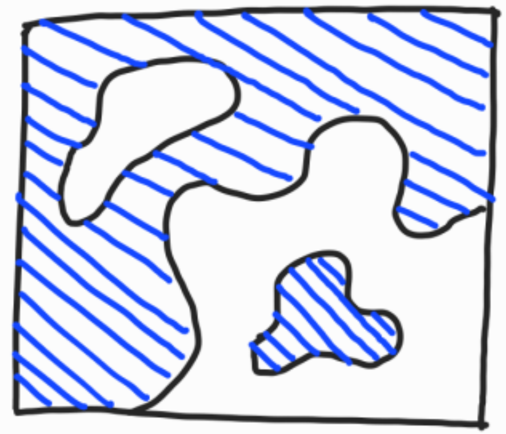

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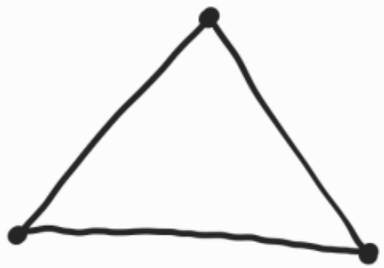

subgraphs

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$a,\ a+b,\ a+2b$

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solutions to linear equations

ARITHMETIC SETTING

Arithmetic Removal (Green 2005, Shapira 2010)

Let $\mathcal{L} = (L_1, \ldots, L_k)$ be a linear system.
For all $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $A \subseteq \mathbb{F}_p^n$ and $\Lambda_{\mathcal{L}}(A) < \delta$, then A can be made $\mathcal{L}$-free by removing $\leq \varepsilon |\mathbb{F}_p^n|$ elements from A.

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$$\Lambda_{\mathcal{L}}(A) = \mathbb{E}_{\underline{x} \in (\mathbb{F}_p^n)^\ell} \mathbb{1}_A(L_1(\underline{x})) \dots \mathbb{1}_A(L_k(\underline{x}))$$

— density of $\mathcal{L}$-instances

ARITHMETIC SETTING

(Linear) Arithmetic Regularity (Green 2005)

For every $\varepsilon > 0$ and $f : \mathbb{F}_p^n \to [0,1]$,
$\exists$ subspace $H \leq \mathbb{F}_p^n$ of codimension $\leq C(\varepsilon)$
s.t. f is ε-Fourier uniform on all but
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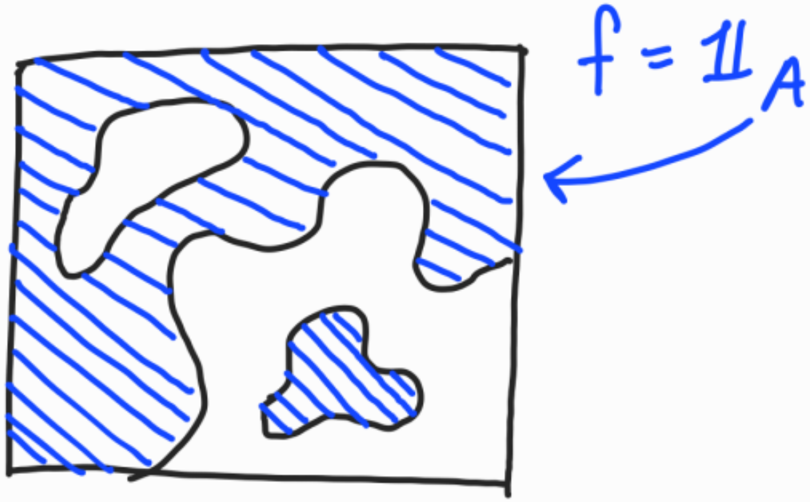

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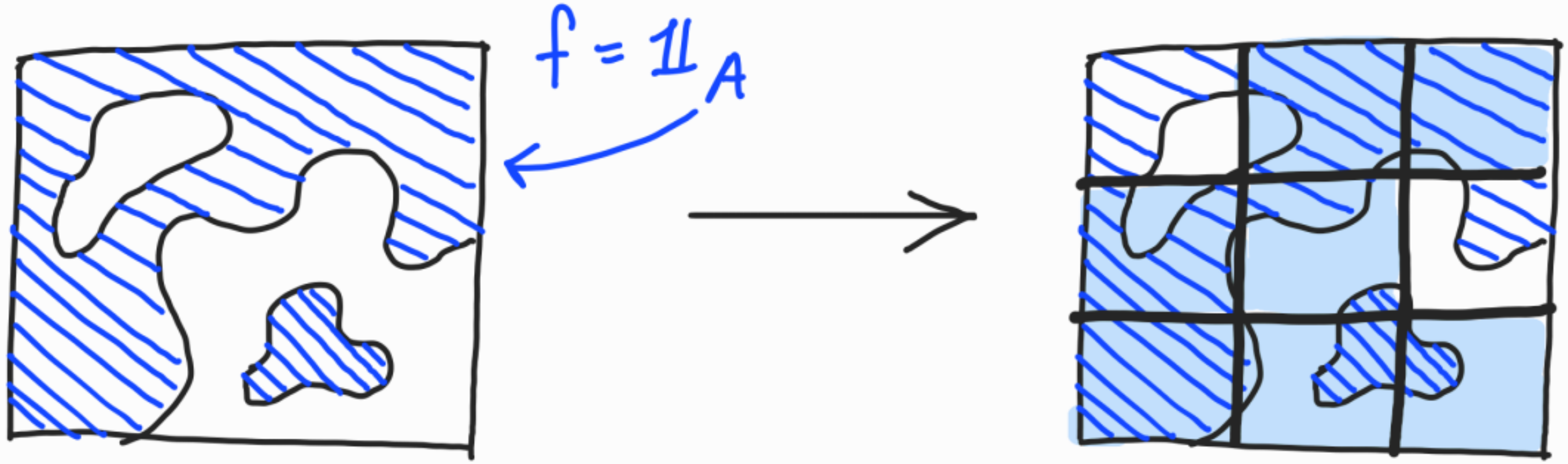

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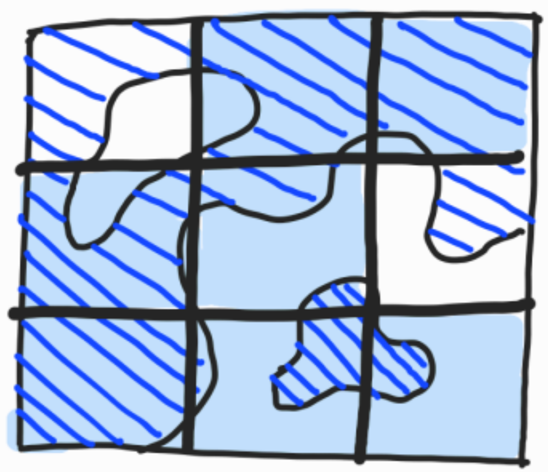
Counting lemma
for linear systems of
true complexity 1



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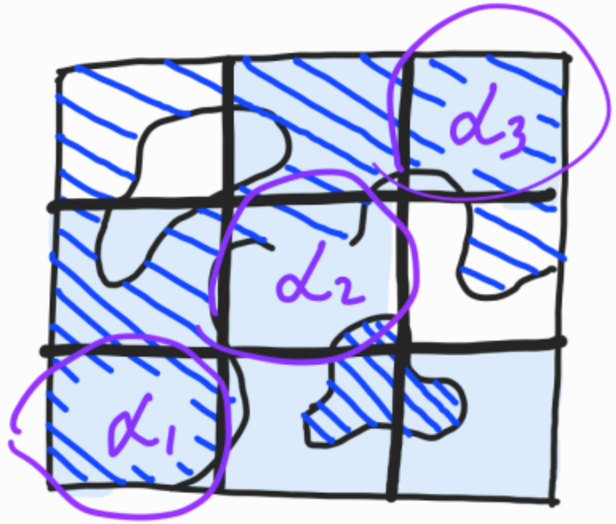

$$\#\text{3-APs} \approx \alpha_1 \alpha_2 \alpha_3 |H|^2$$

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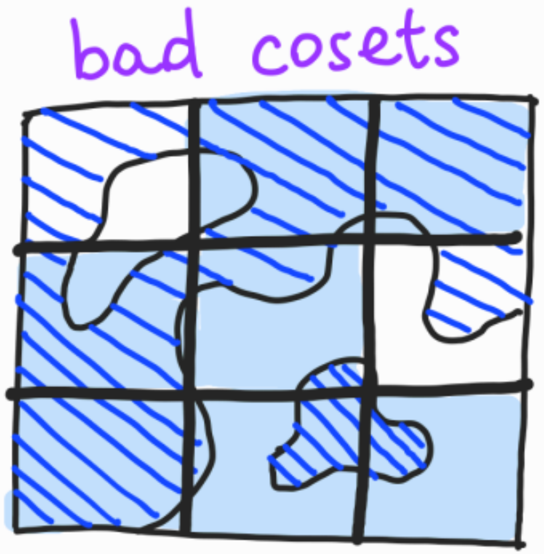

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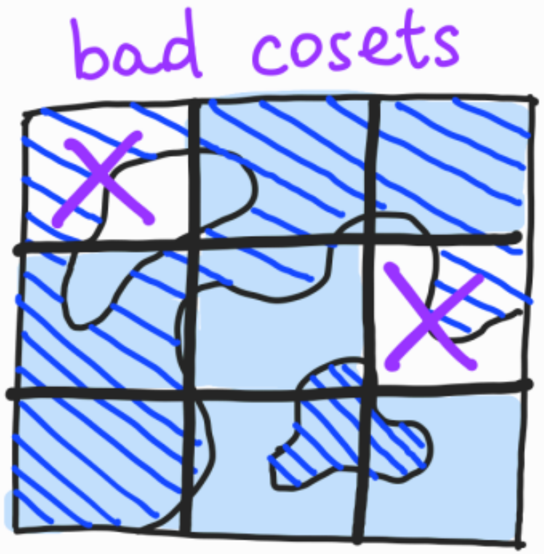

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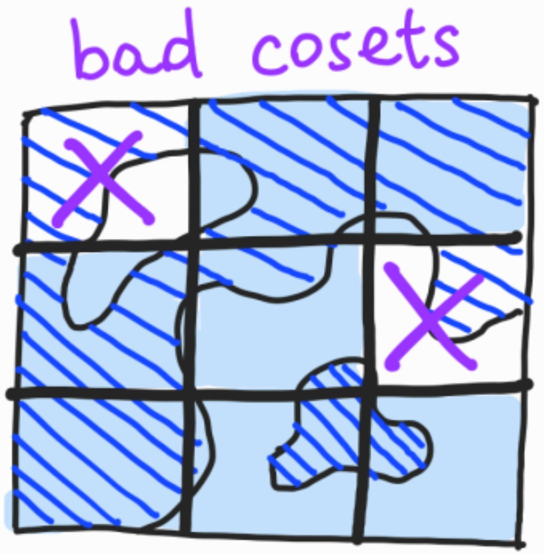

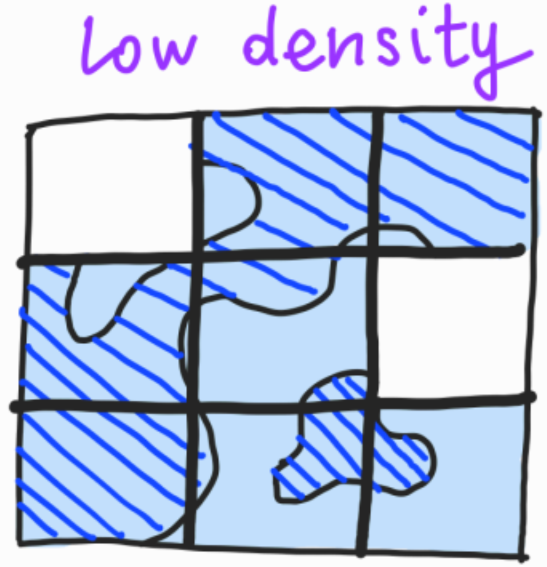

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Arithmetic Removal (Green 2005, Shapira 2010)

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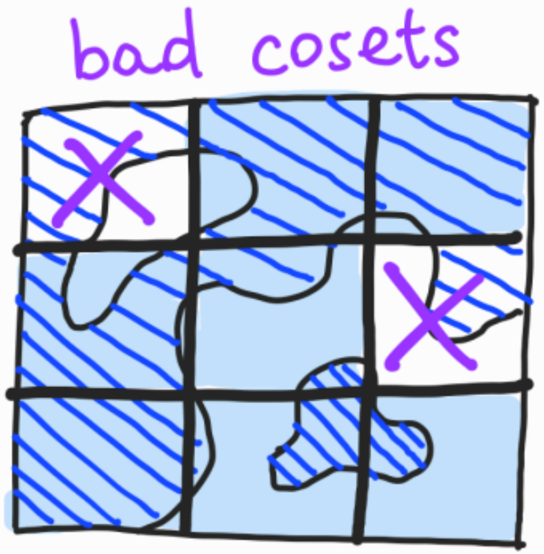

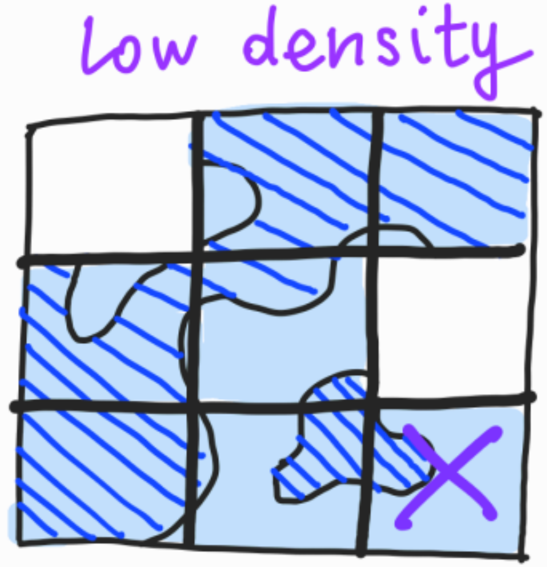

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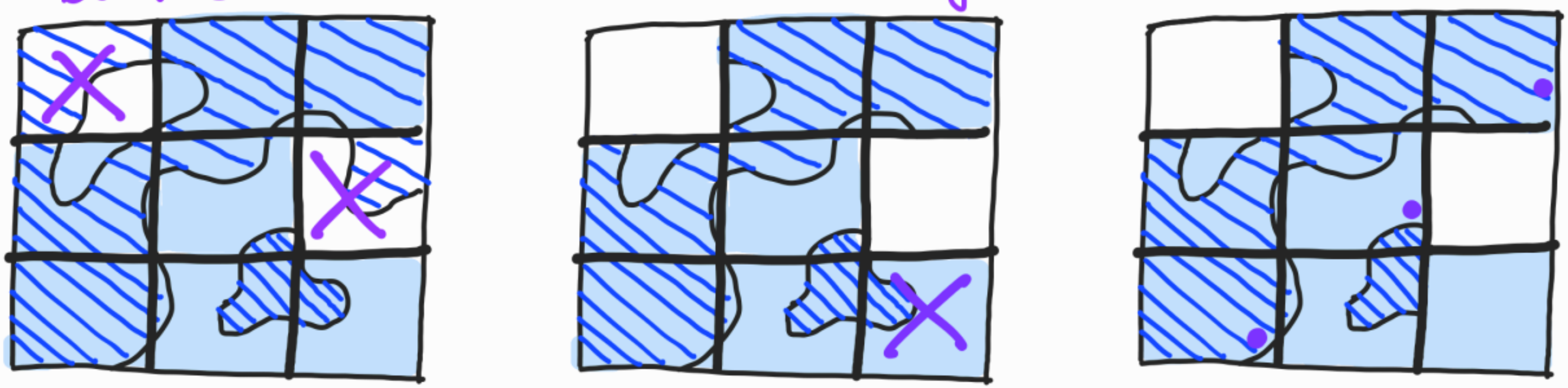
bad cosets

low density



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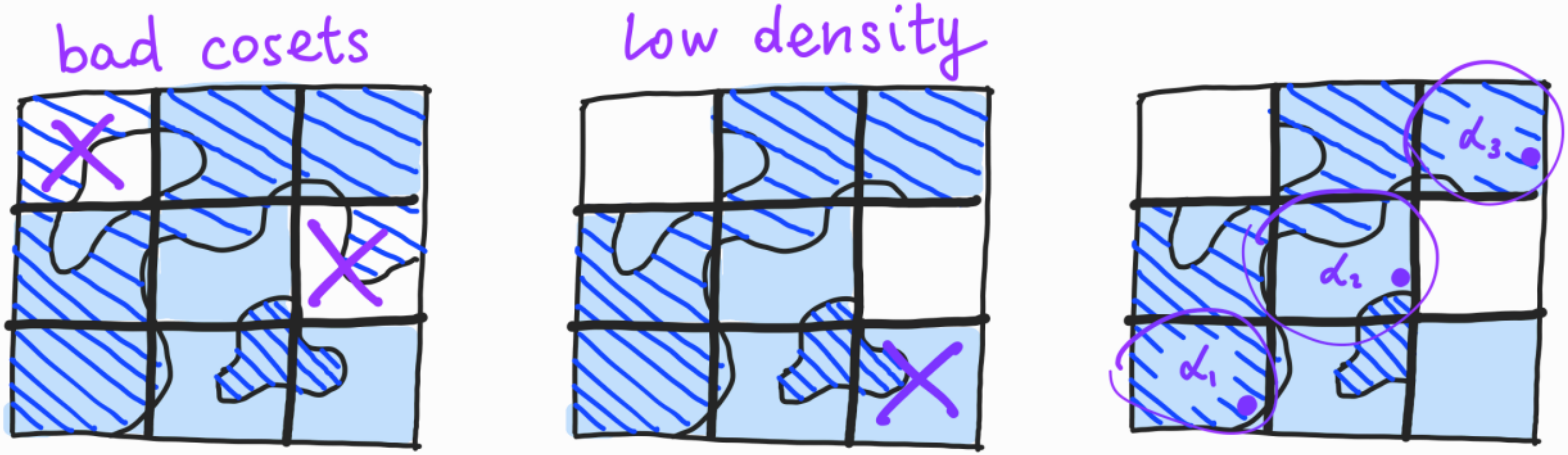

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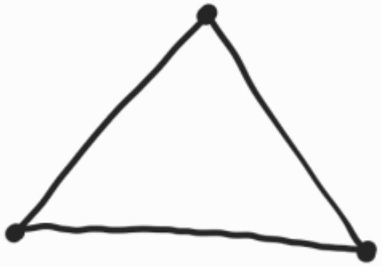
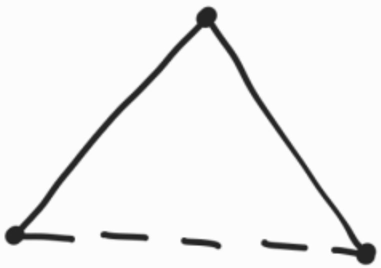


subgraphs

$\longrightarrow$

$$x - 2y + z = 0$$

solutions to linear equations <u>in a set</u>



induced subgraphs

$\longrightarrow$

$$x - 2y + z = 0$$

coloured solutions under an r-colouring

arithmetic pattern

ARITHMETIC SETTING

Induced Arithmetic Removal (Fox, Tidor, Zhao 2022)

Let $\mathcal{H}$ be an arithmetic pattern of true complexity 1. For all $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $\chi : \mathbb{F}_p^n \to [r]$ satisfies $\Lambda_{\mathcal{H}}(\chi) < \delta$, then χ can be made $\mathcal{H}$-free on $\mathbb{F}_p^n \setminus \{0\}$ by recolouring $\leq \varepsilon |\mathbb{F}_p^n|$ elements.

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$$\Lambda_{\mathcal{H}}(\chi) = \mathop{\mathbb{E}}_{\underline{x} \in (\mathbb{F}_p^n)^\ell} 1\!\!1_R(x)\, 1\!\!1_B(x+d)\, 1\!\!1_B(x+2d)$$

$$R = \chi^{-1}(\text{red})$$

$$\mathcal{H} = (\underline{x},\ \underline{x+d},\ \underline{x+2d})$$

$$B = \chi^{-1}(\text{teal})$$

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- Tidor, Zhao 2022 : any complexity (more exceptions)
- Bhattacharyya, Fischer, H.+P. Hatami, Lovett 2013 :
  translation-invariant patterns, <u>no</u> exceptions

ARITHMETIC SETTING

<u>Partition-regular:</u>
every colouring has a monochromatic solution in $\mathbb{F}_p^n \setminus \{0\}$

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STRONG ARITHMETIC REGULARITY

(Bhattacharyya, Fischer, Lovett 2013)

For every $\varepsilon : \mathbb{N} \to (0,1)$ and $f : \mathbb{F}_p^n \to [0,1]$, $\exists$ subspaces $H_2 \leq H_1 \leq \mathbb{F}_p^n$ of codimensions $\leq C(\varepsilon)$ $D_2 \geq D_1$ such that:

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Pythagoras' Theorem

$$\mathcal{E}(H_2) - \mathcal{E}(H_1) = \mathbb{E}(\alpha_{H_2+c} - \alpha_{H_1+c})^2$$

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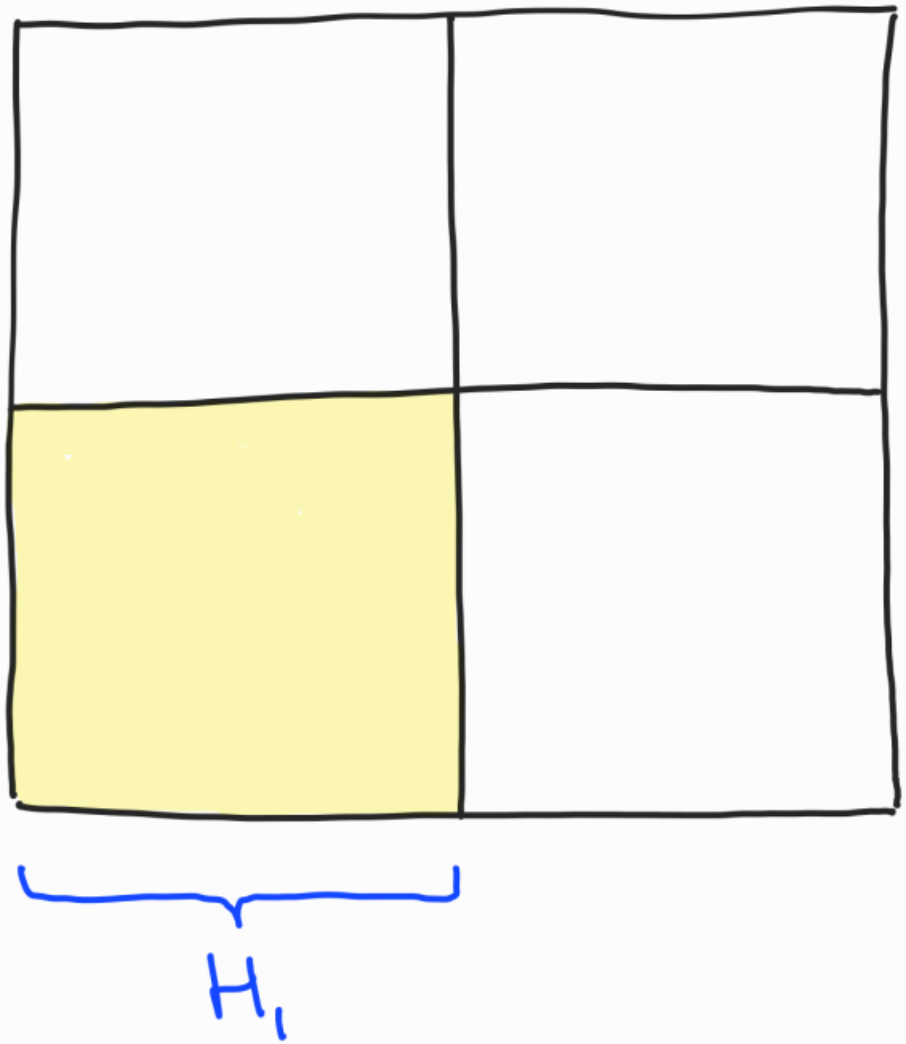

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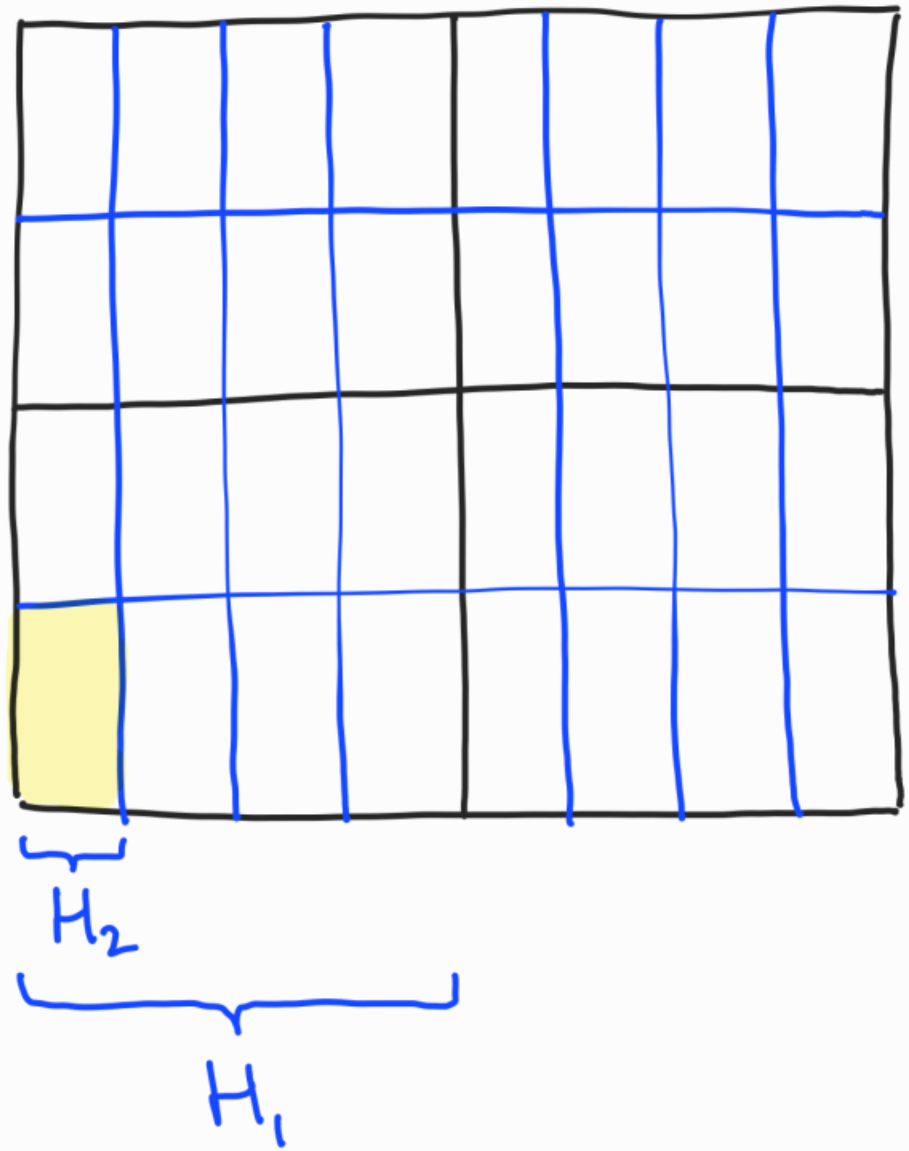

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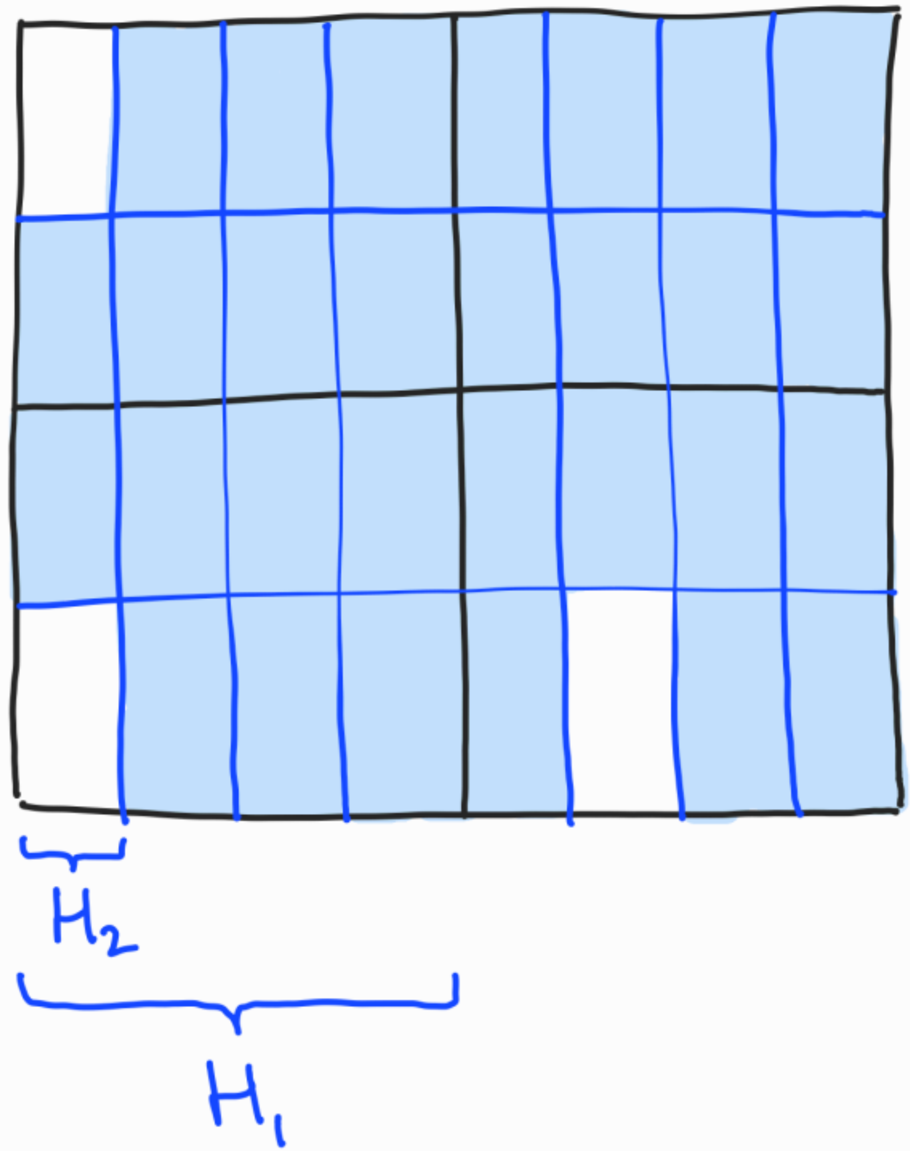

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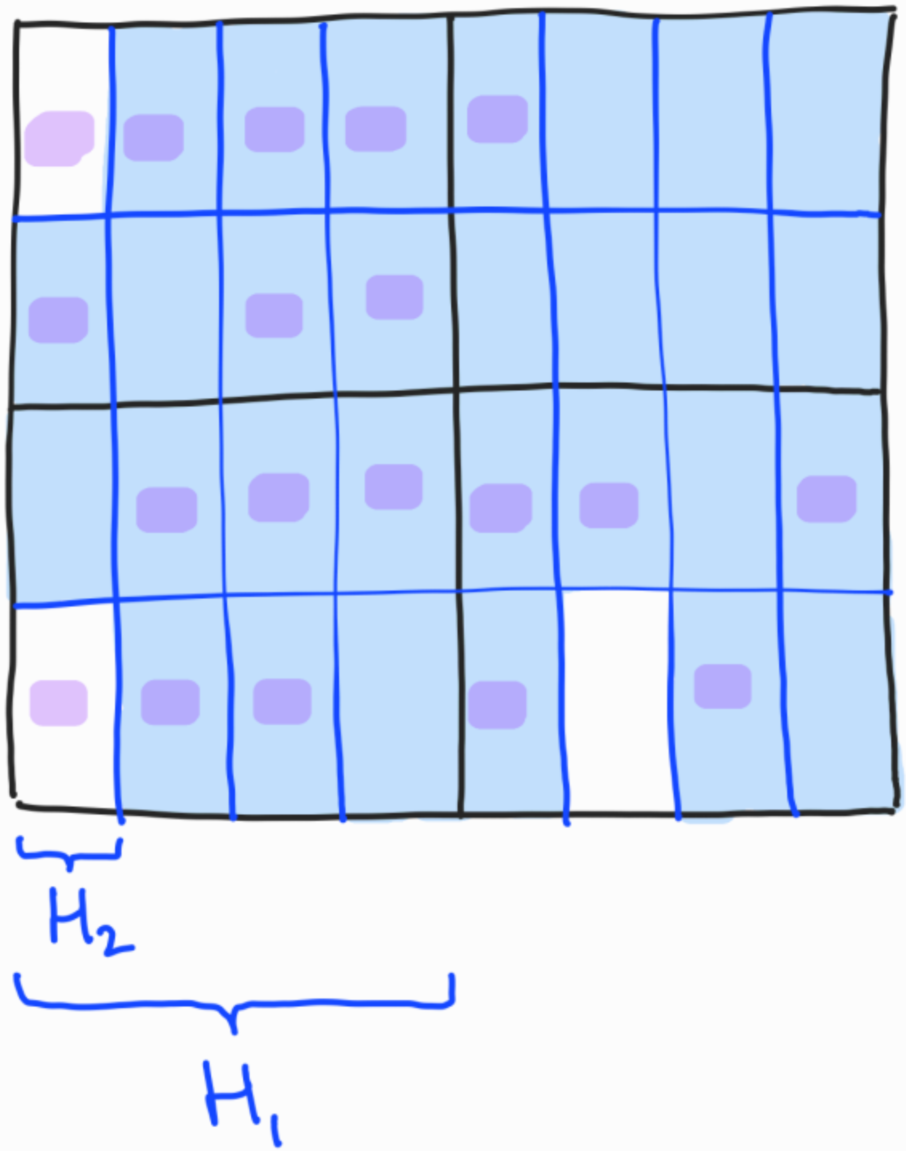

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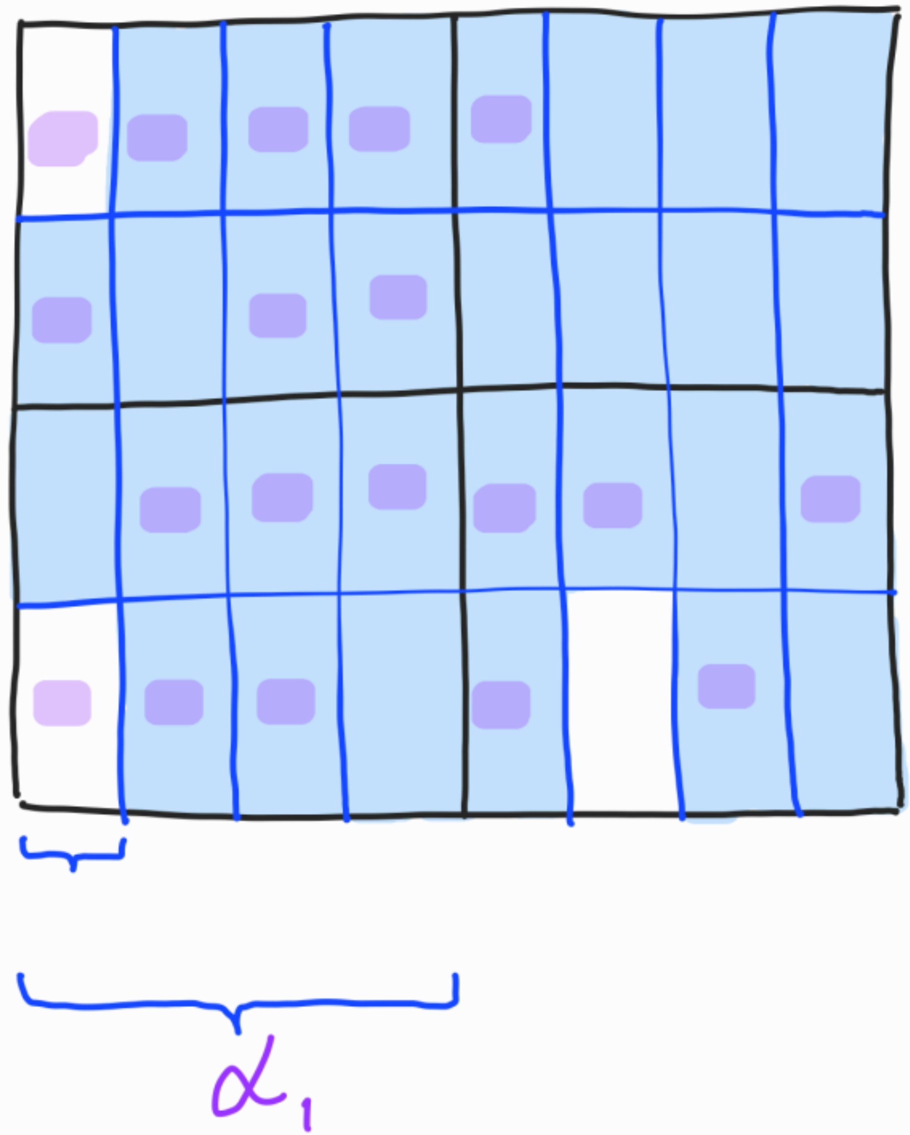

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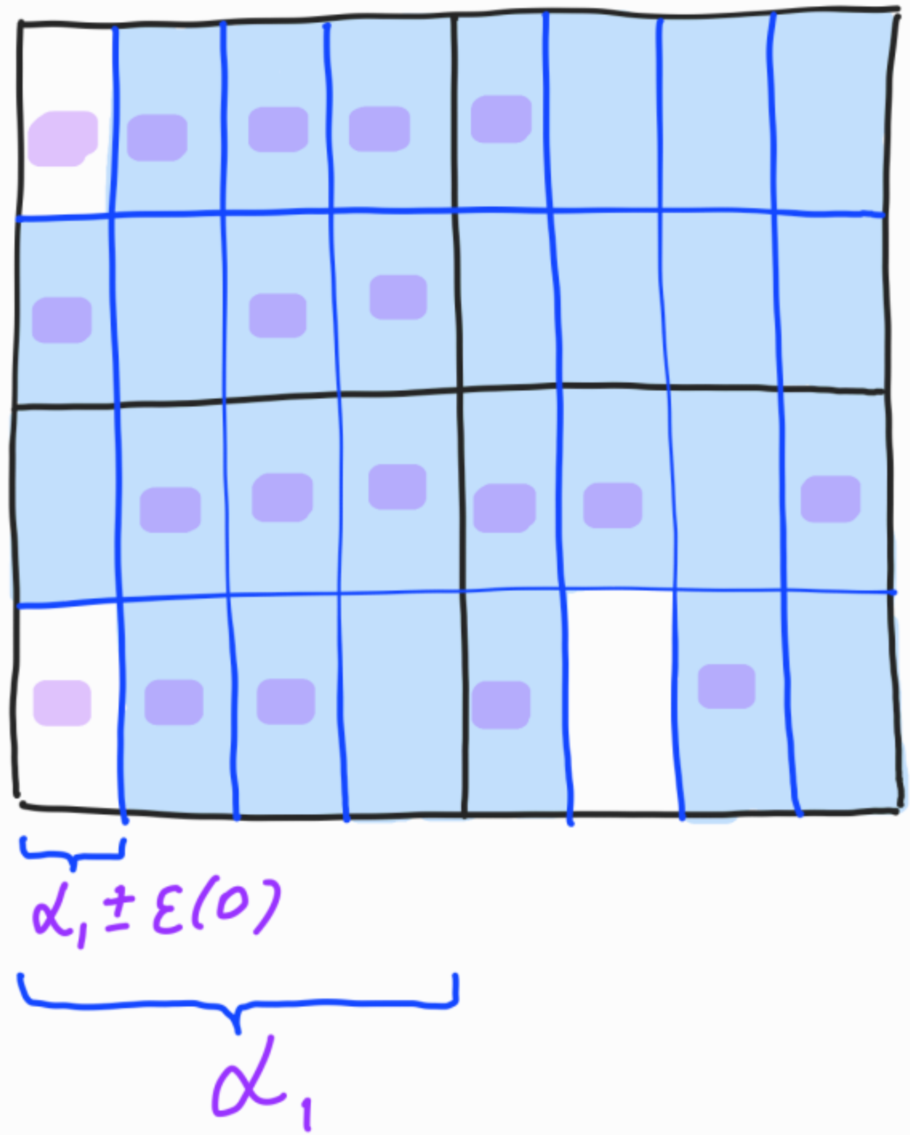

INDUCED ARITHMETIC REMOVAL

Apply SARL to

$$\underbrace{\chi^{-1}(1), \dots, \chi^{-1}(r)}_{\text{colour classes}}$$

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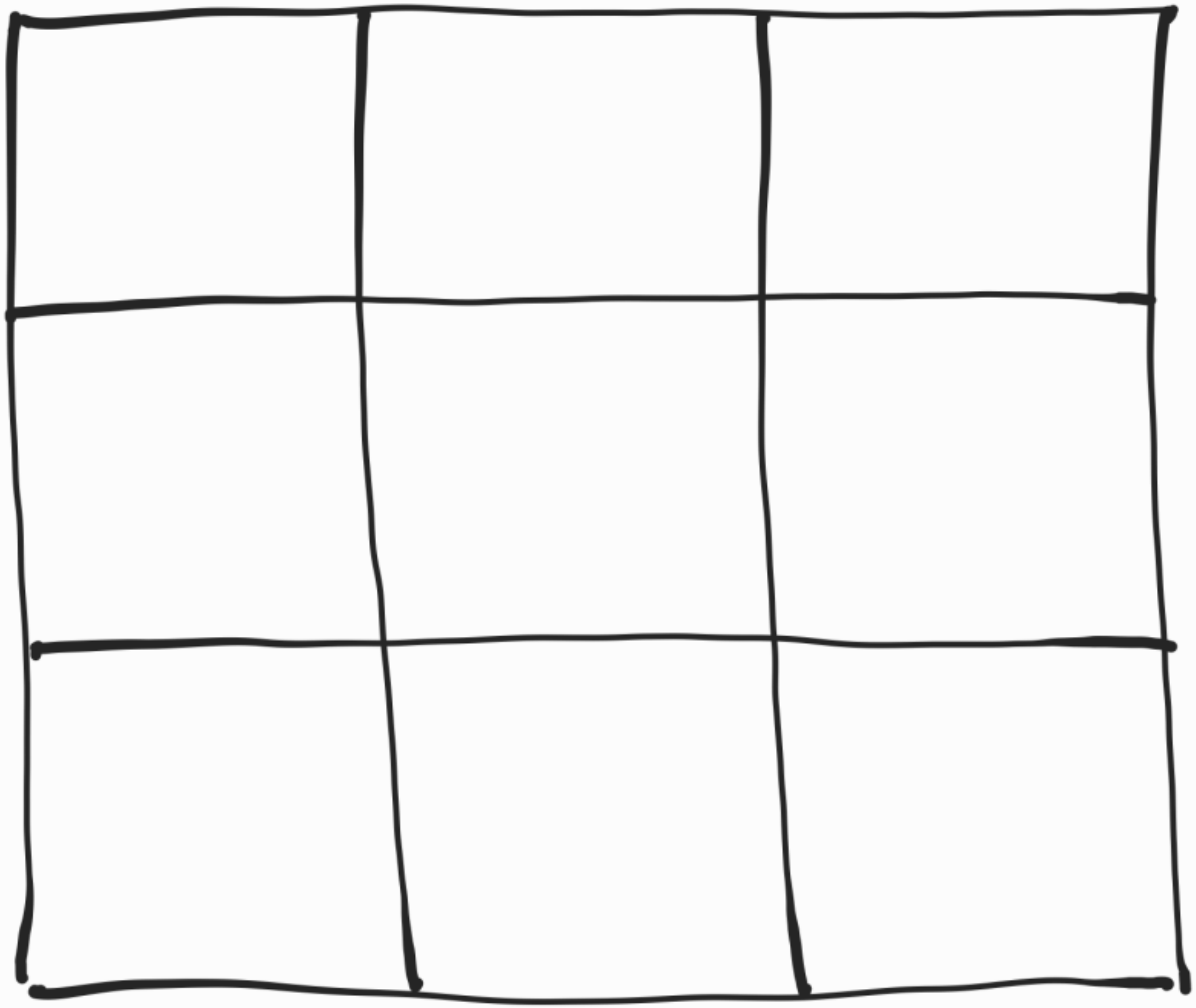

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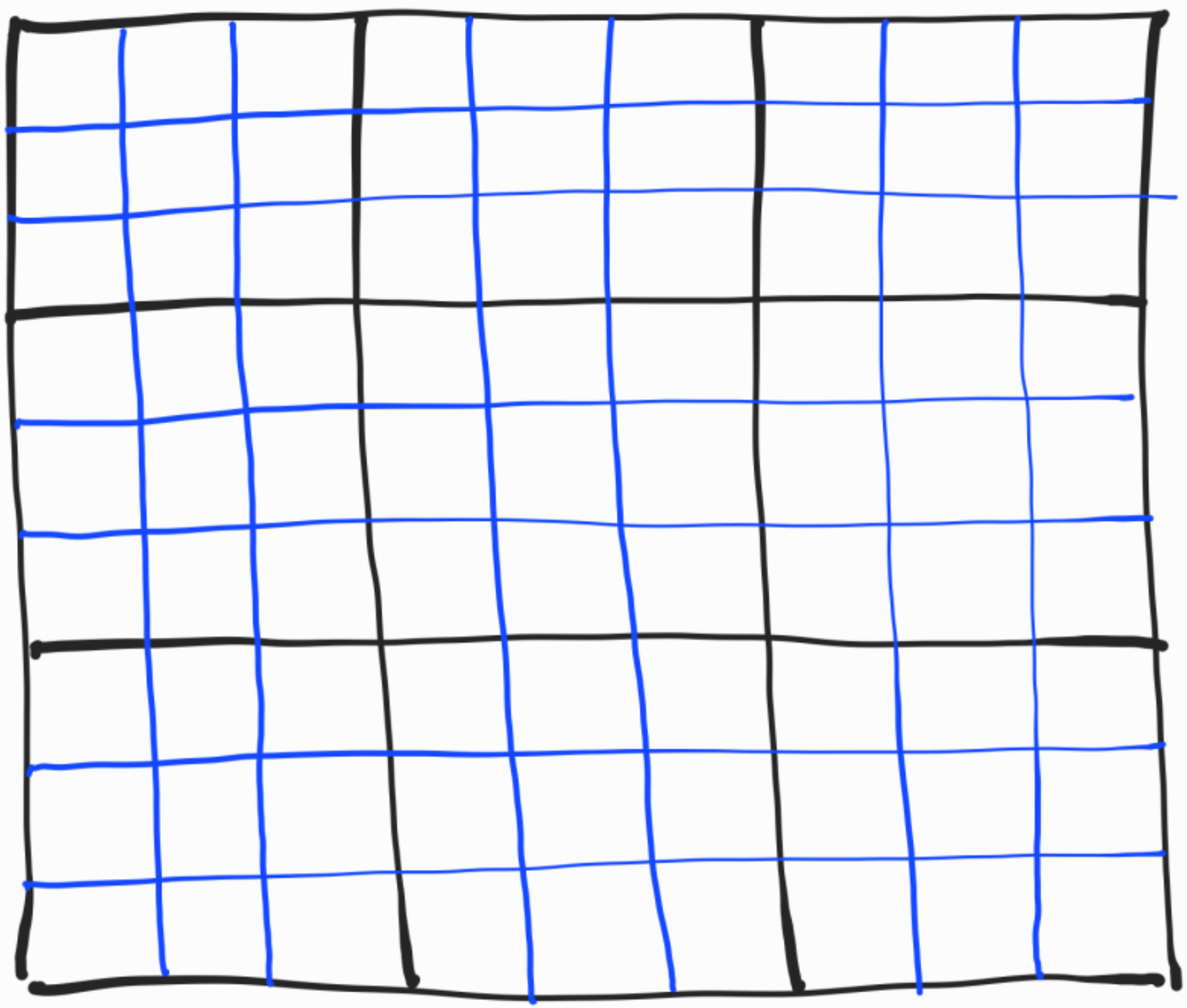

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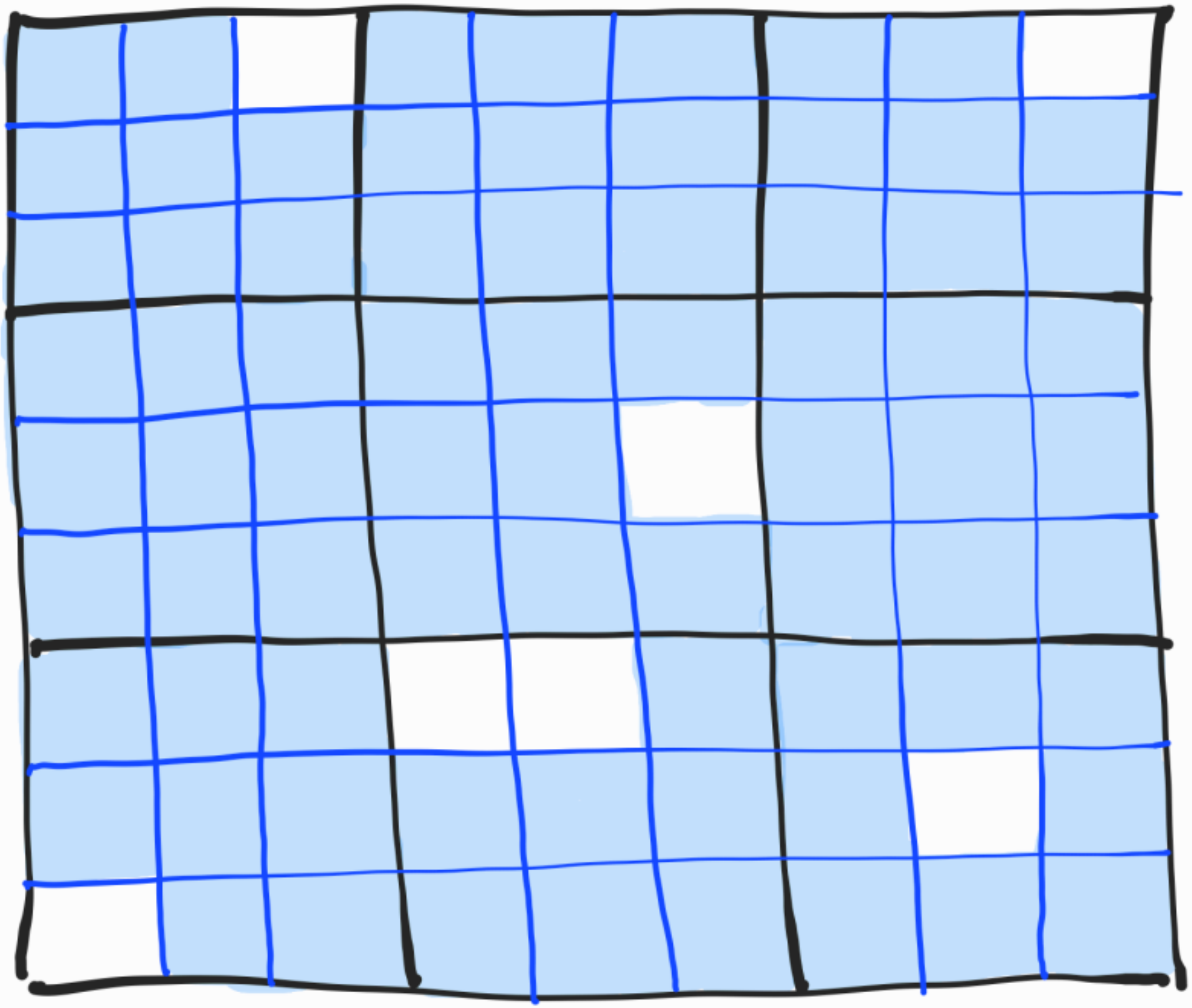

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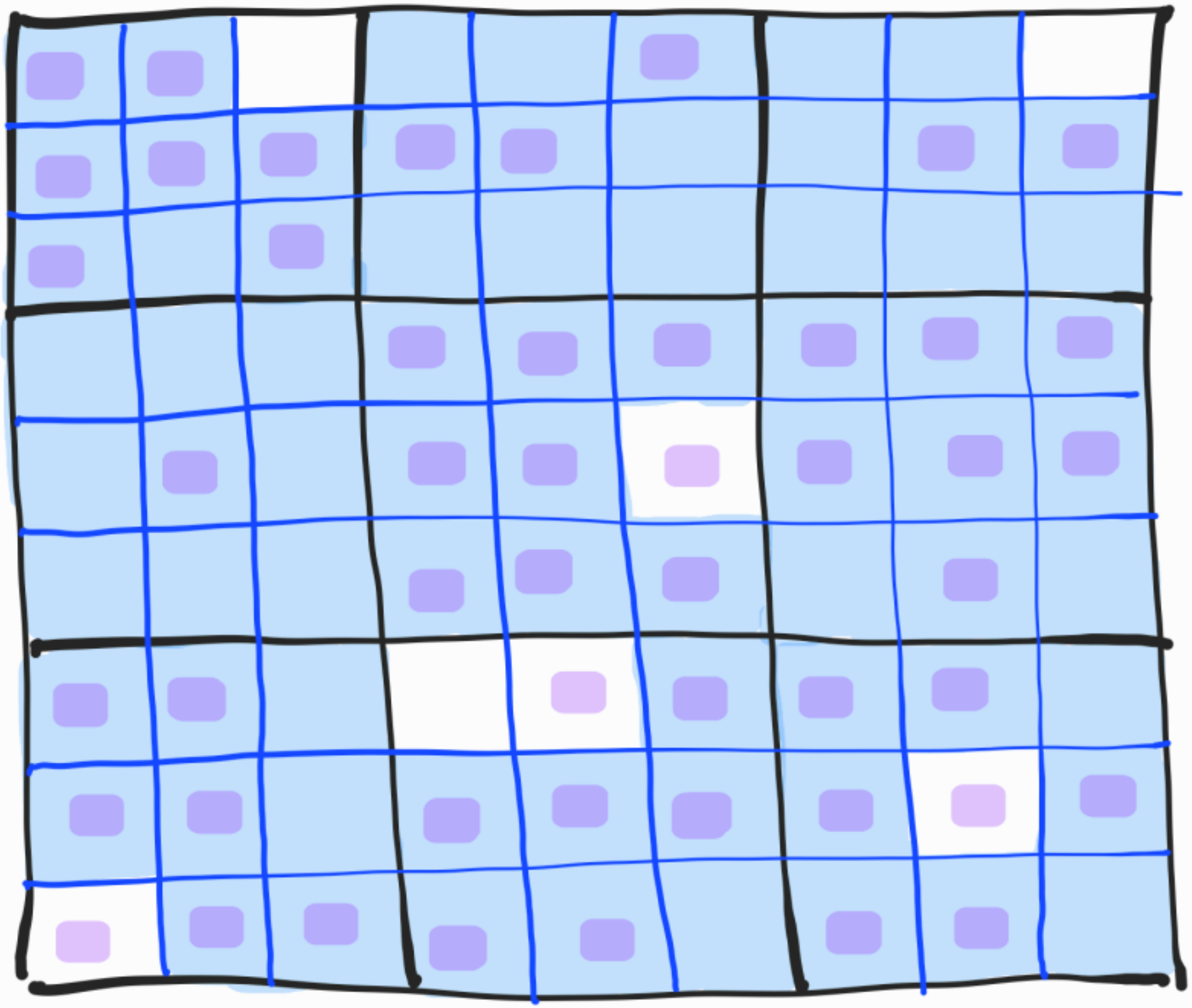

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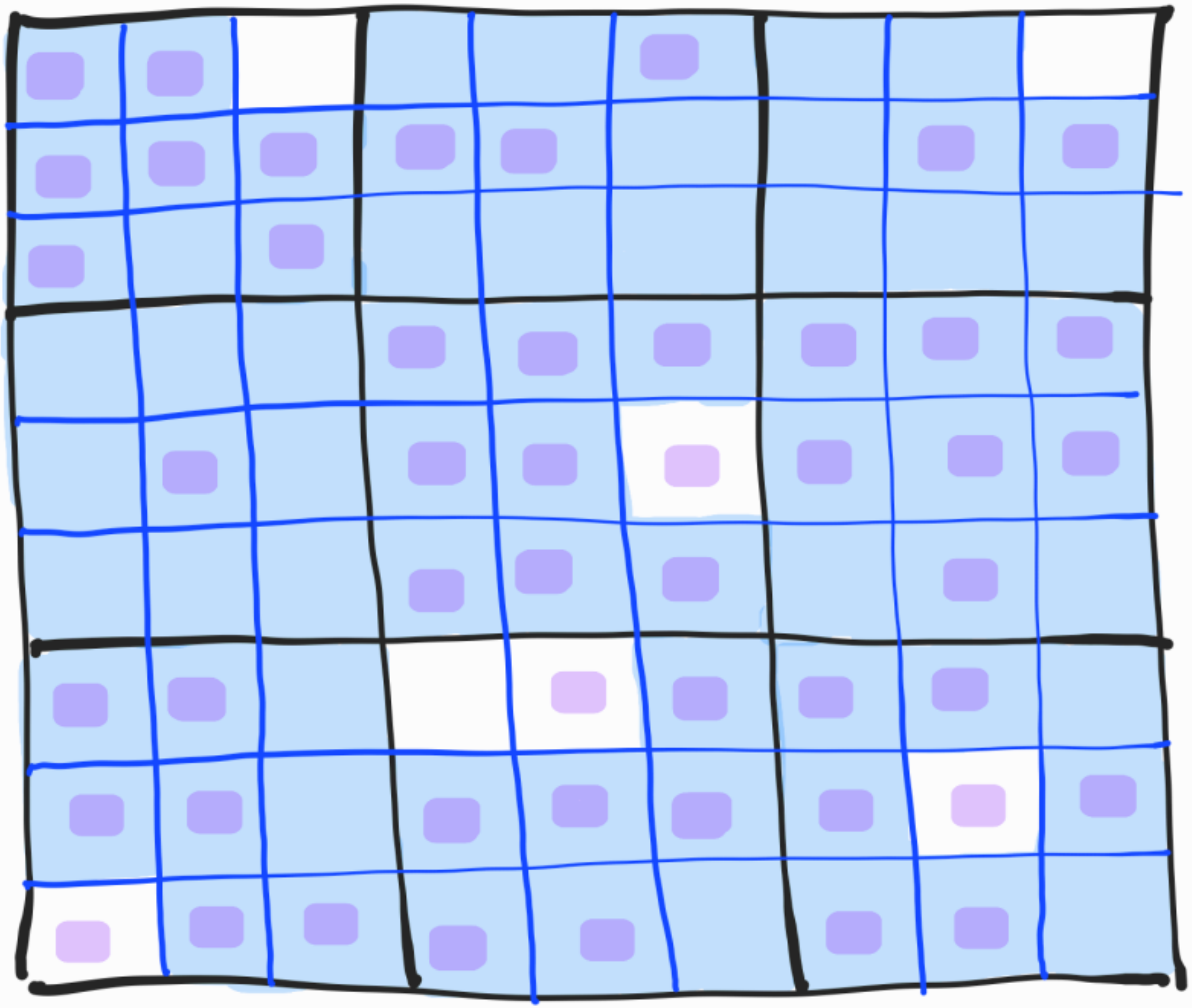

Regular model

Each H_1-coset represented by a regular H_2-coset

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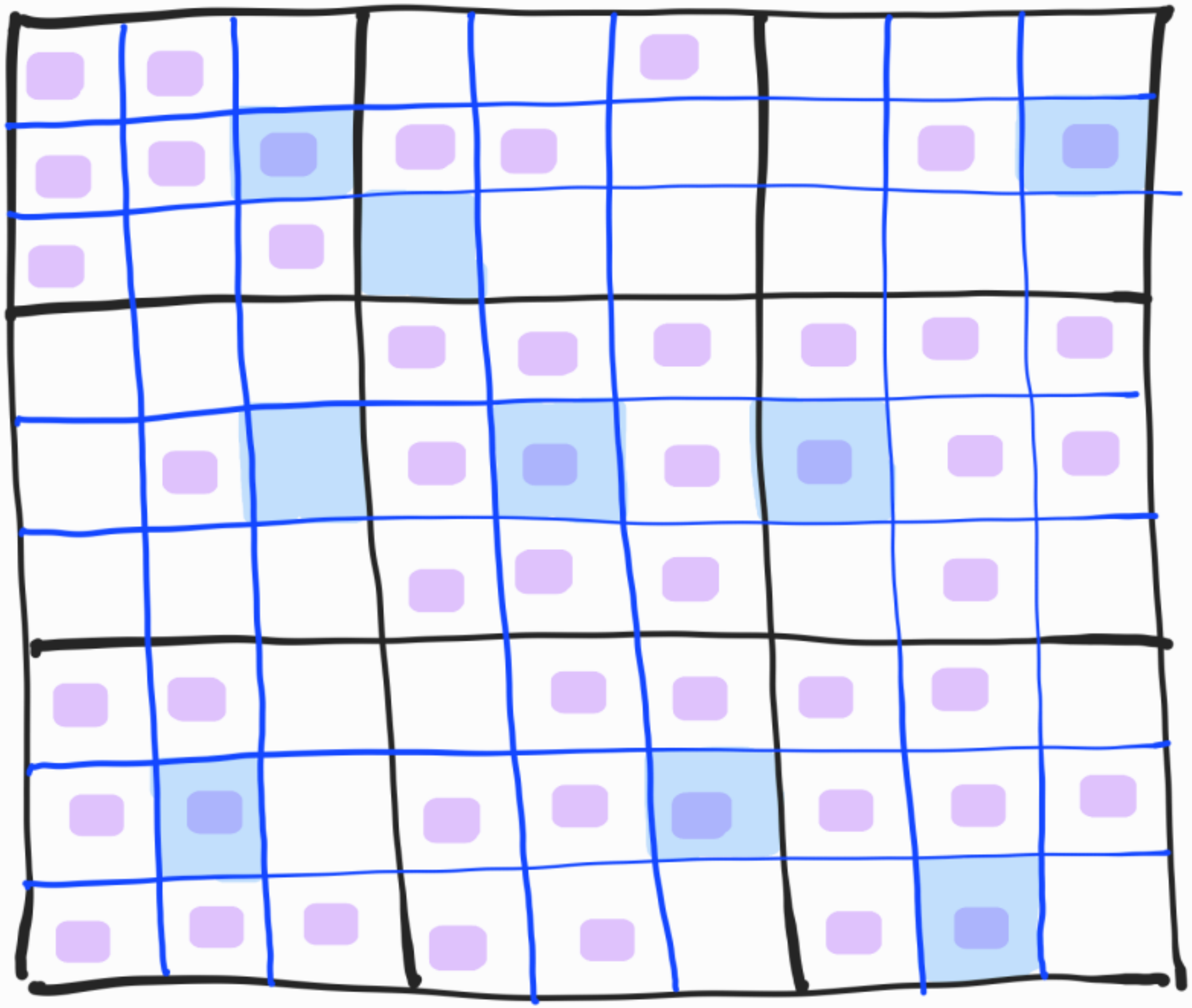

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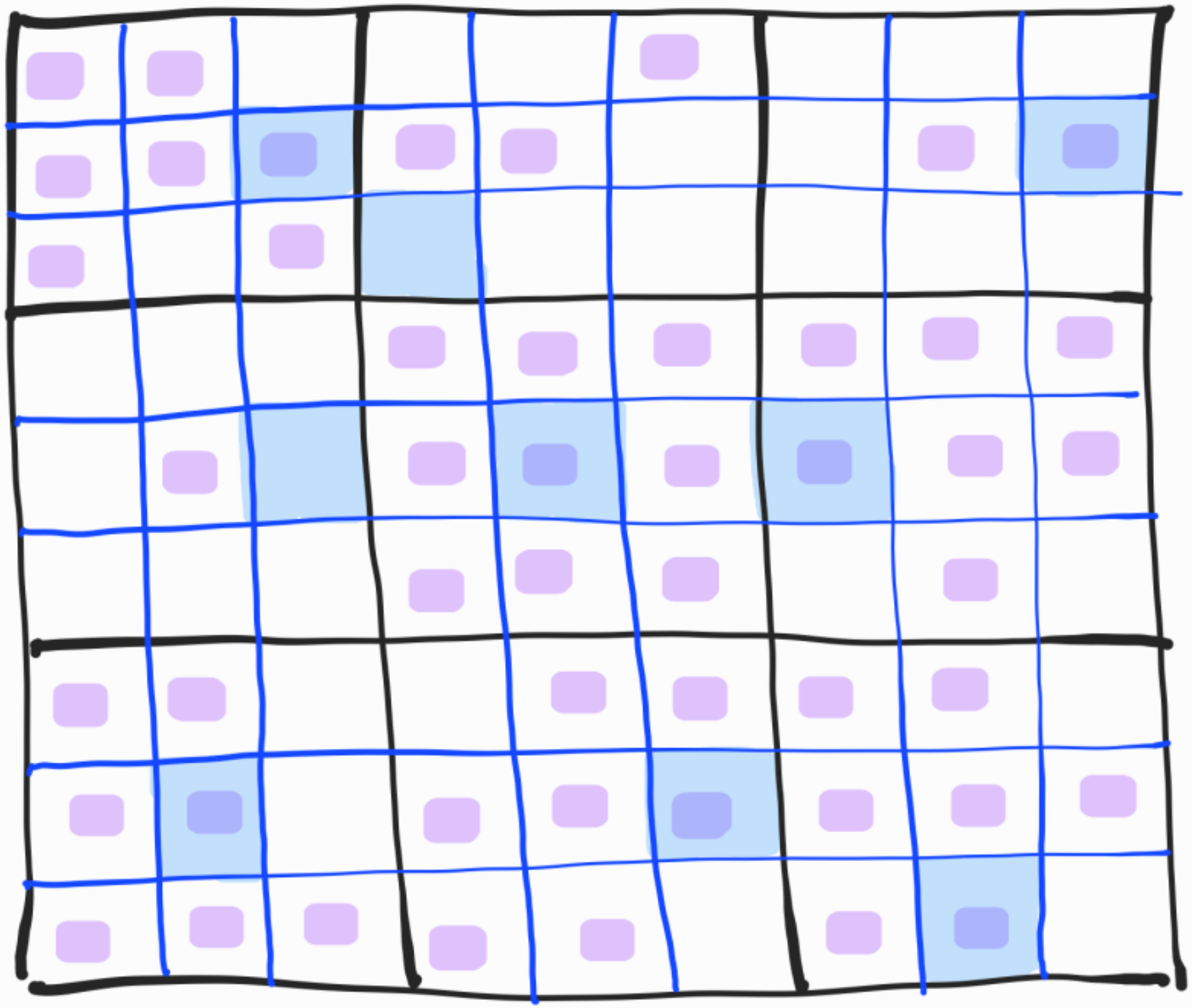

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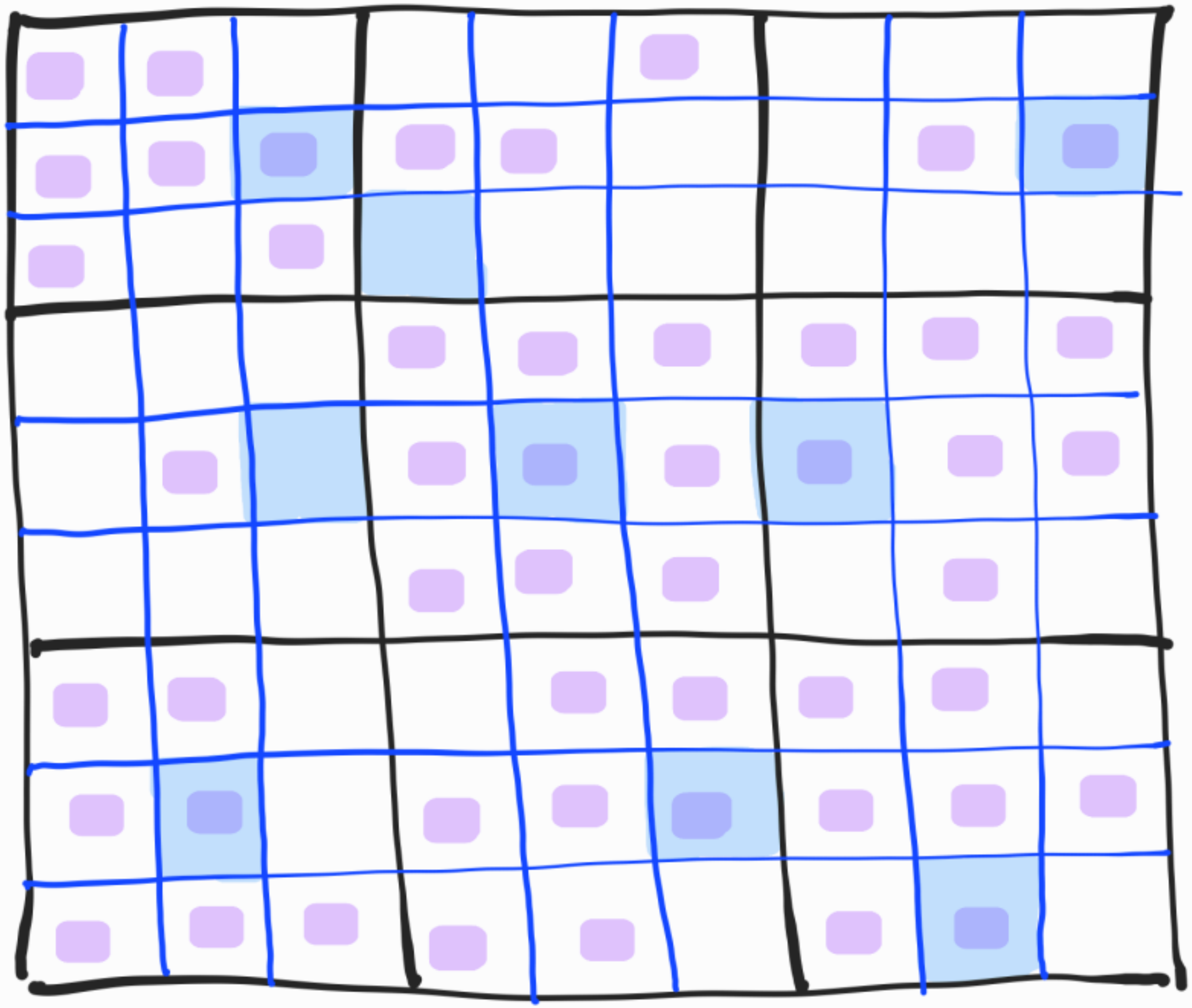
Most representatives "remember" colour densities

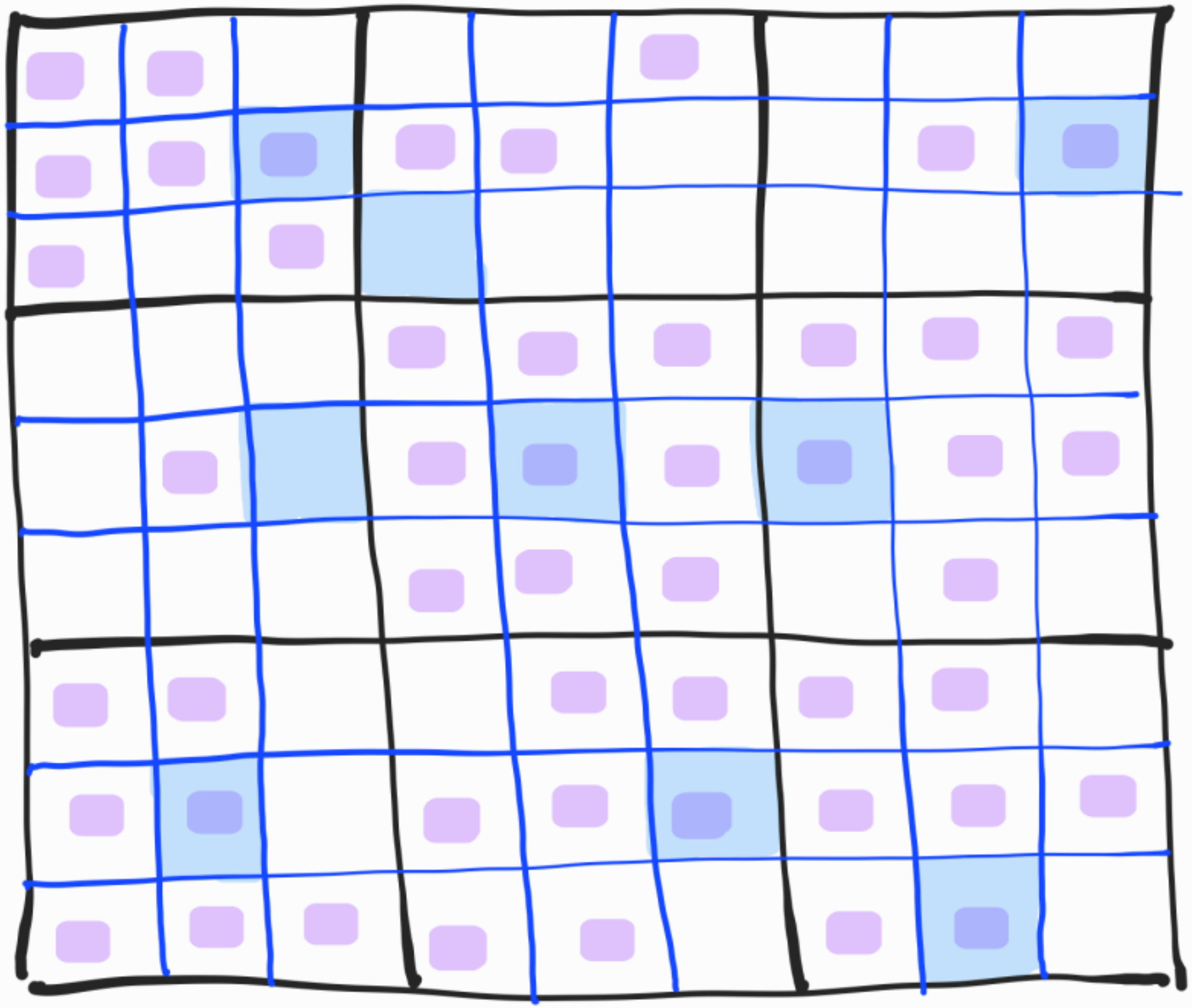
INDUCED ARITHMETIC REMOVAL



Recolouring

Ask the representative

INDUCED ARITHMETIC REMOVAL



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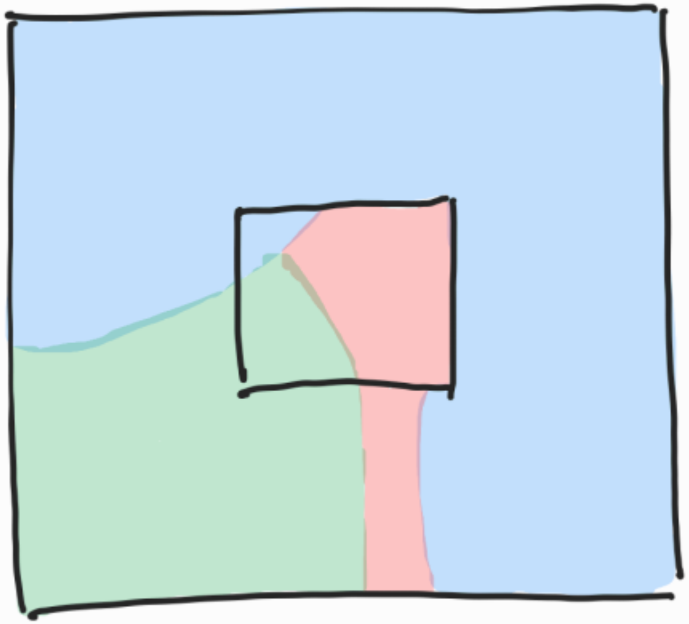

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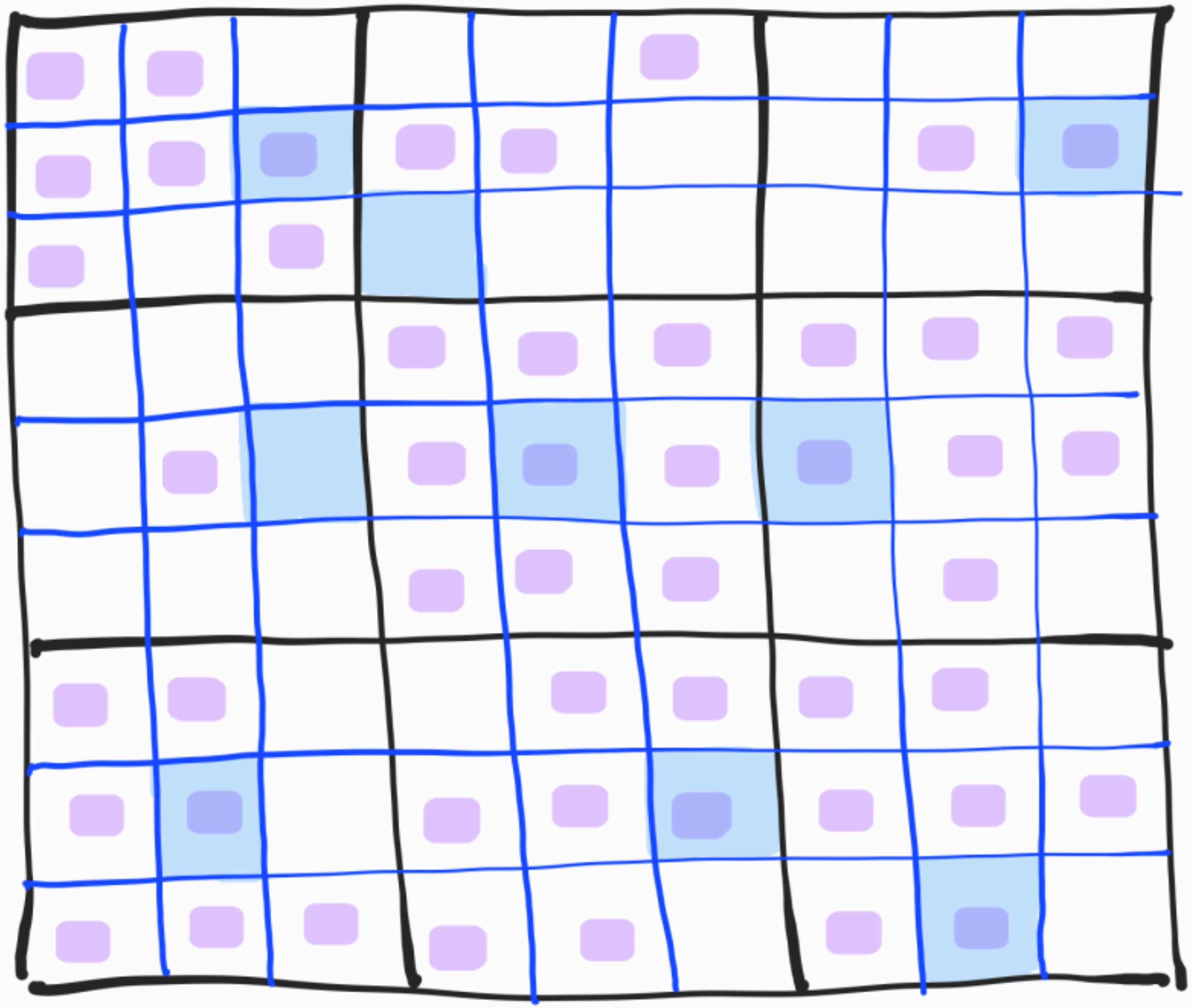
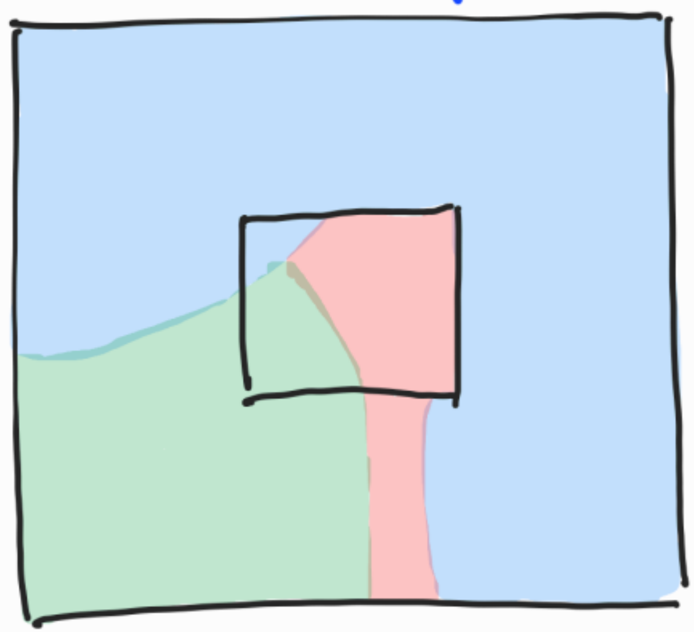


Recolouring

Ask the representative

— high density



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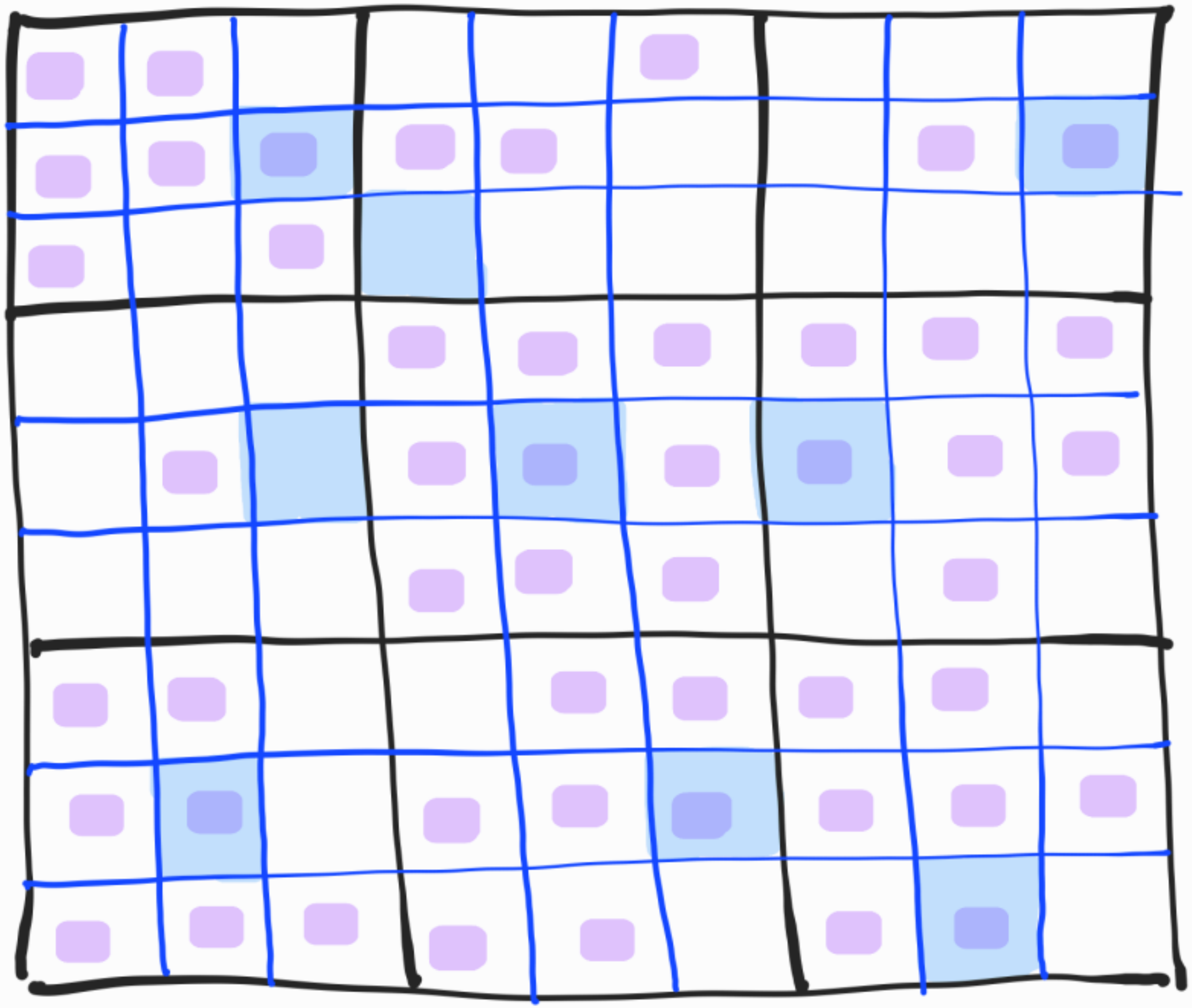

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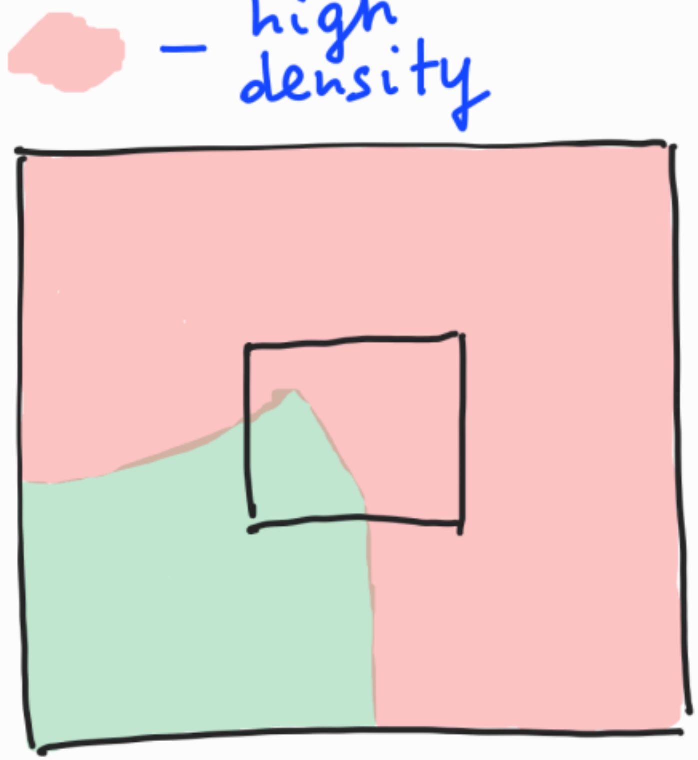

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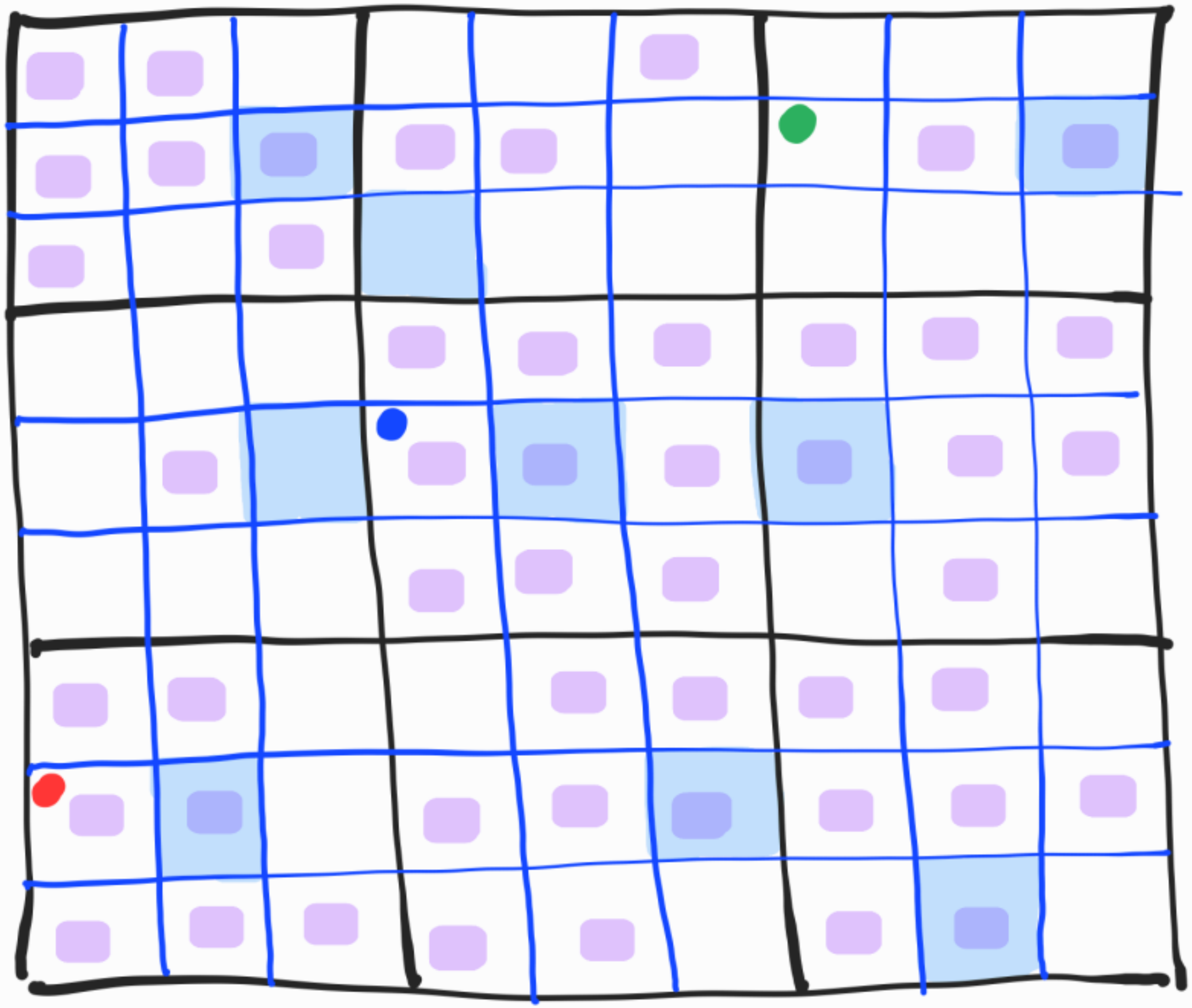

Suppose there is

x, $x + d$, $x + 2d$

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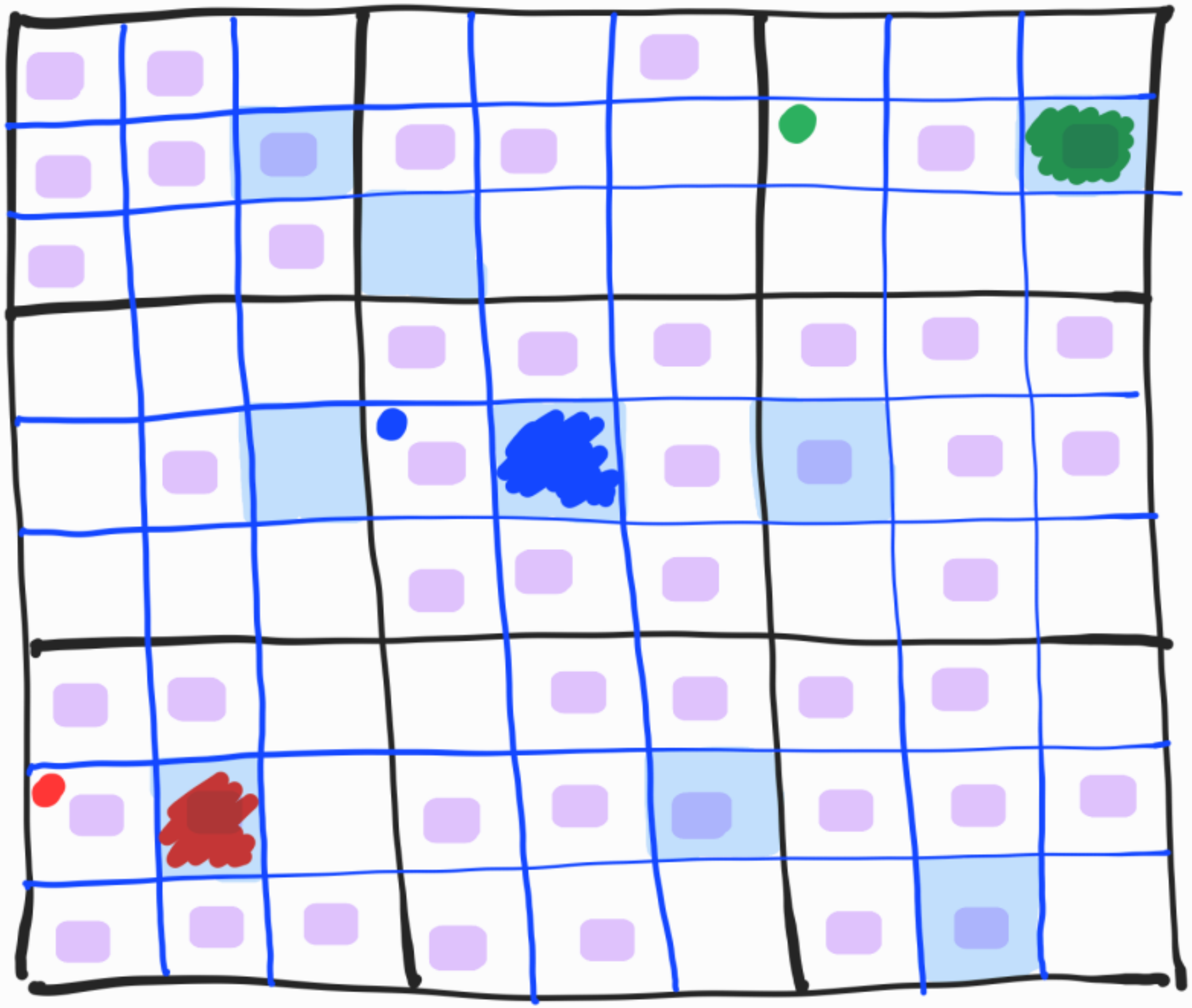

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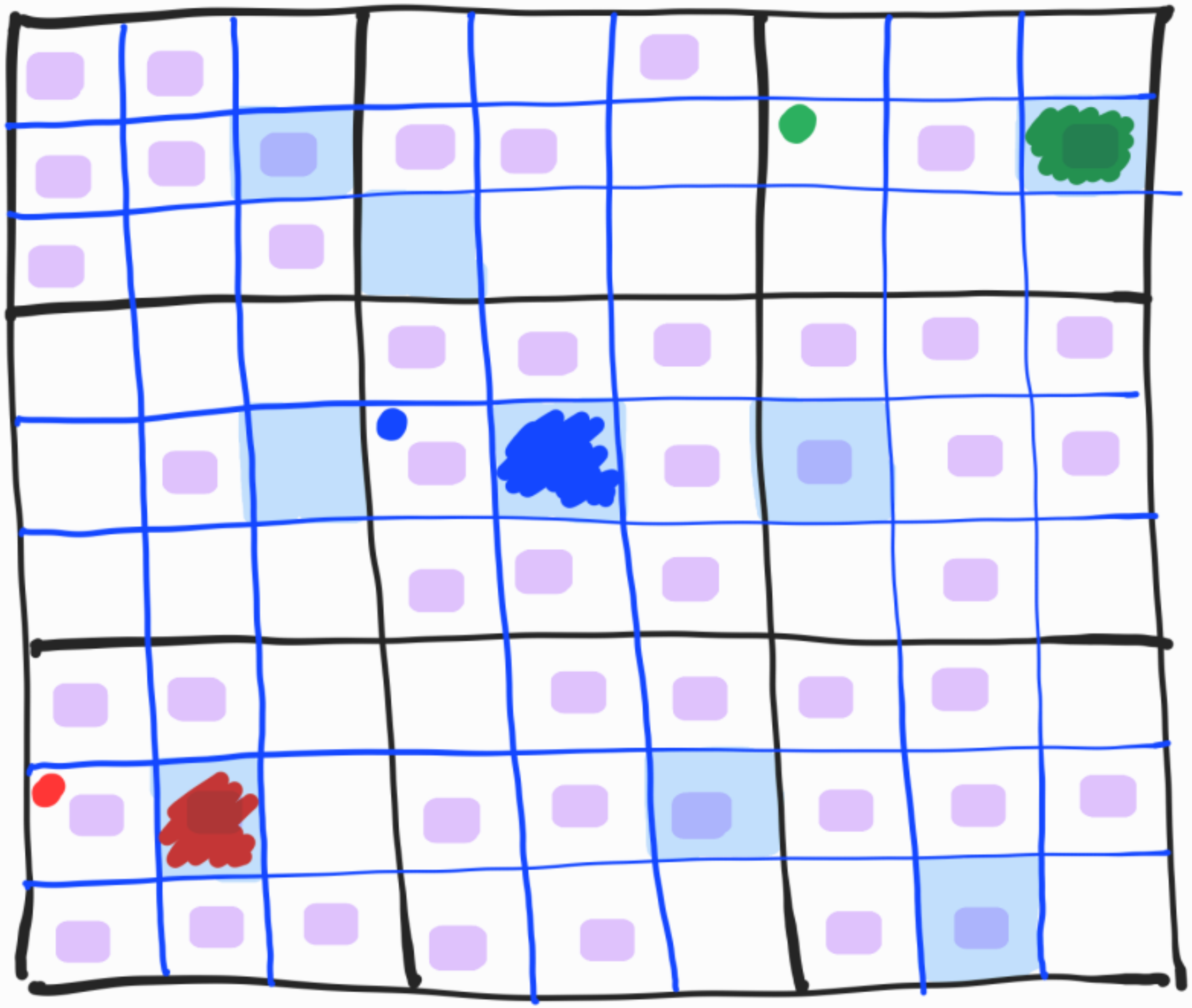

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Would like:

many instances

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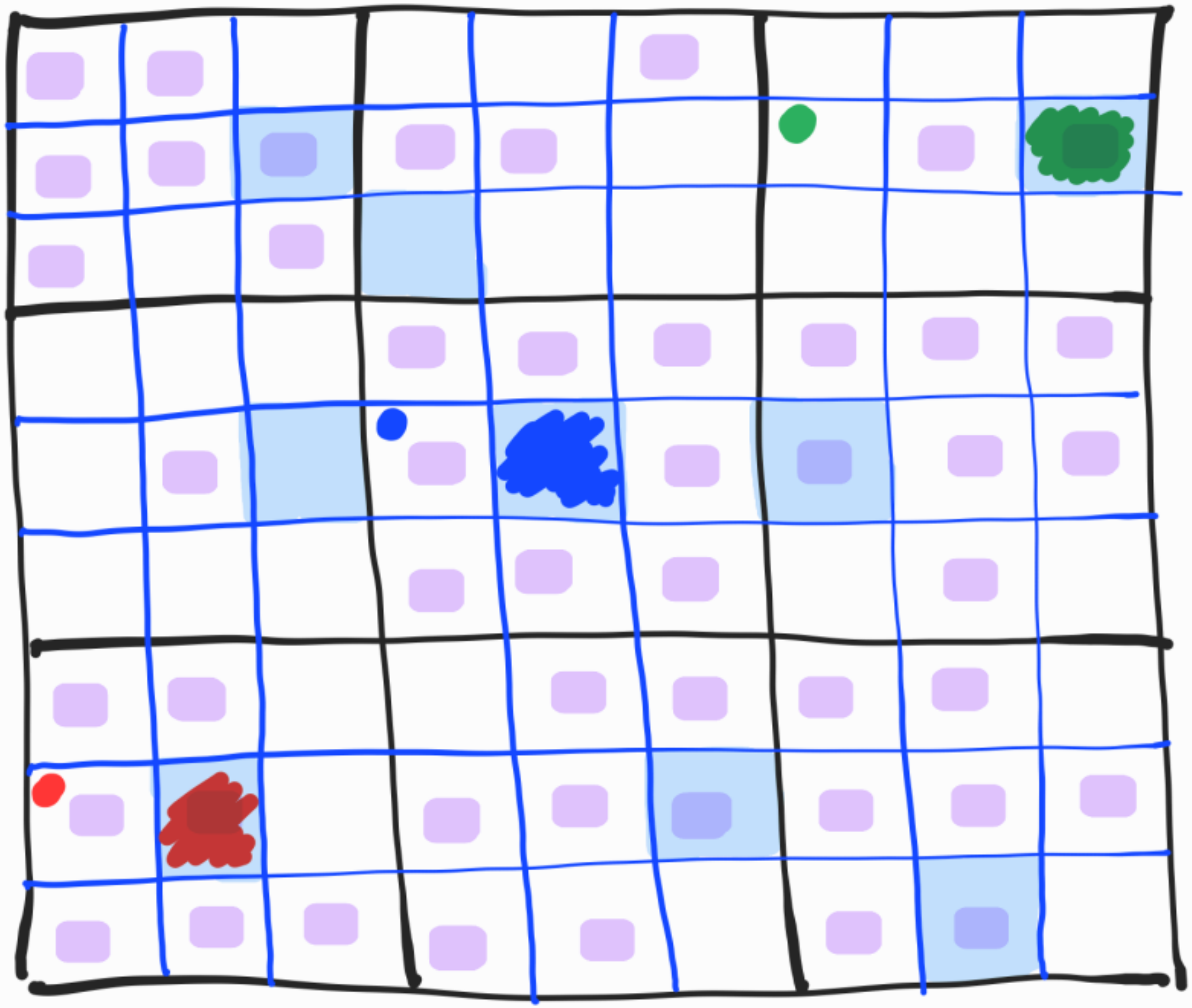

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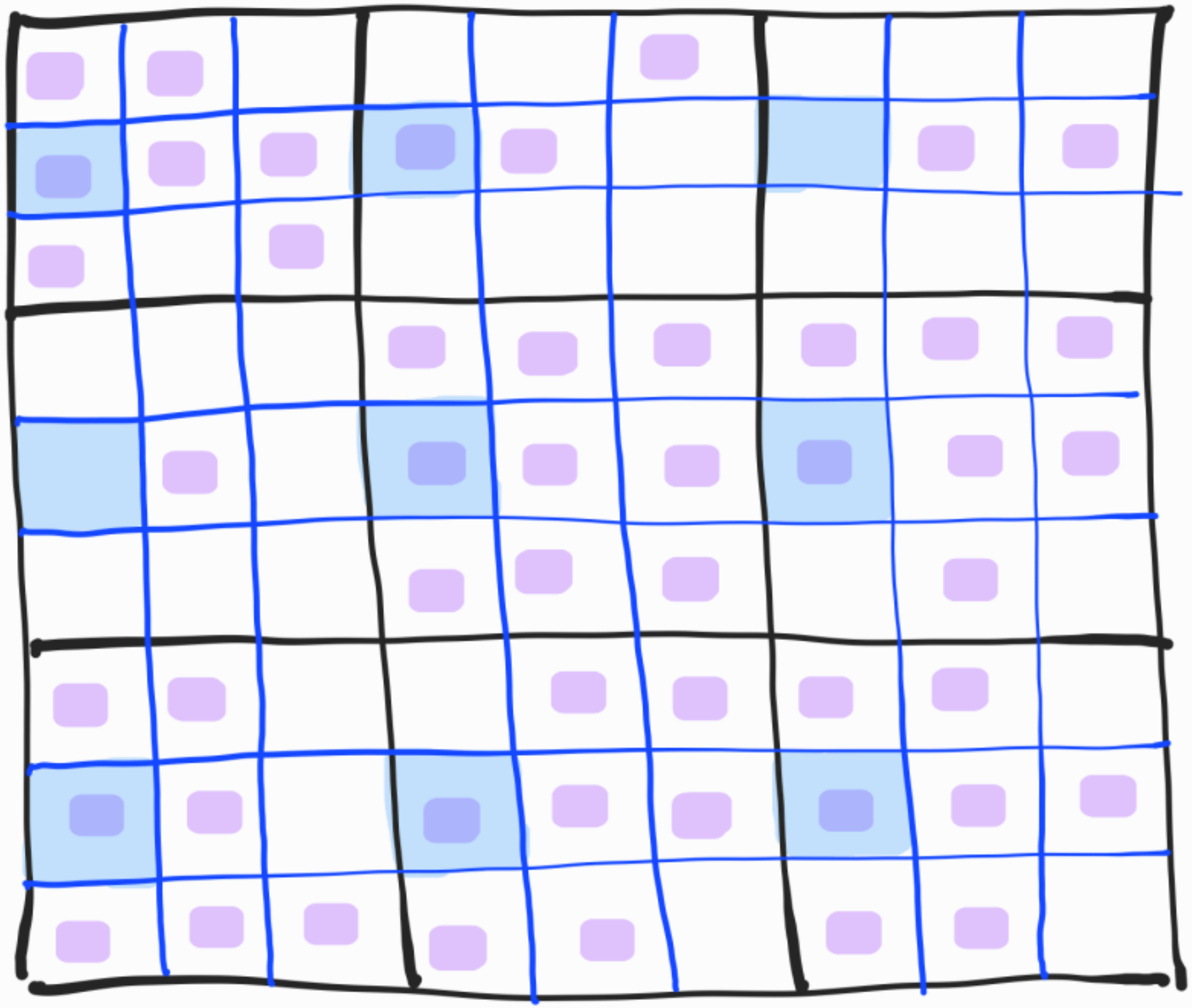
Problem:

subcosets not in a 3-AP!

INDUCED ARITHMETIC REMOVAL



Suppose there is

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Pick subcosets to form an affine subspace

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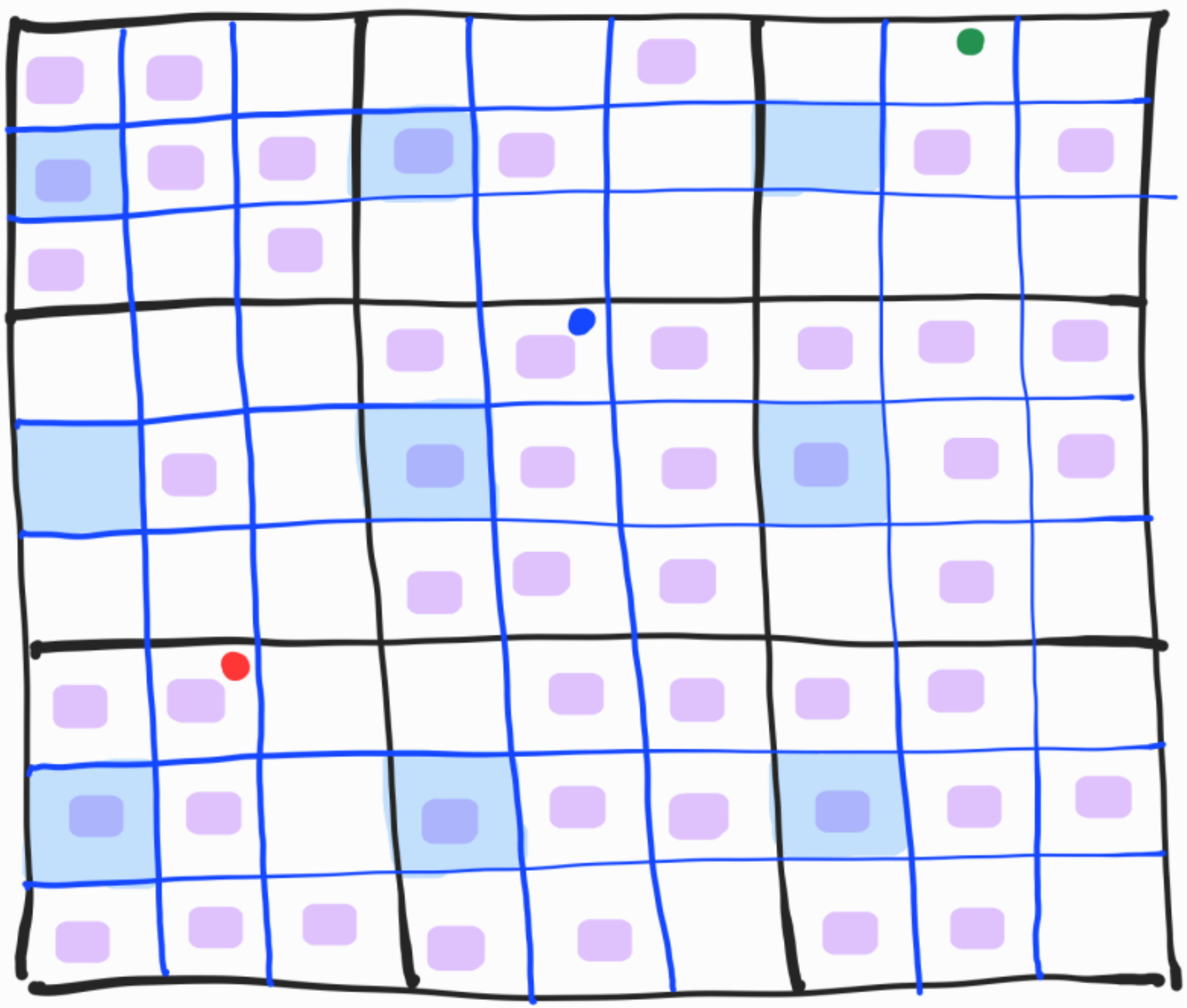

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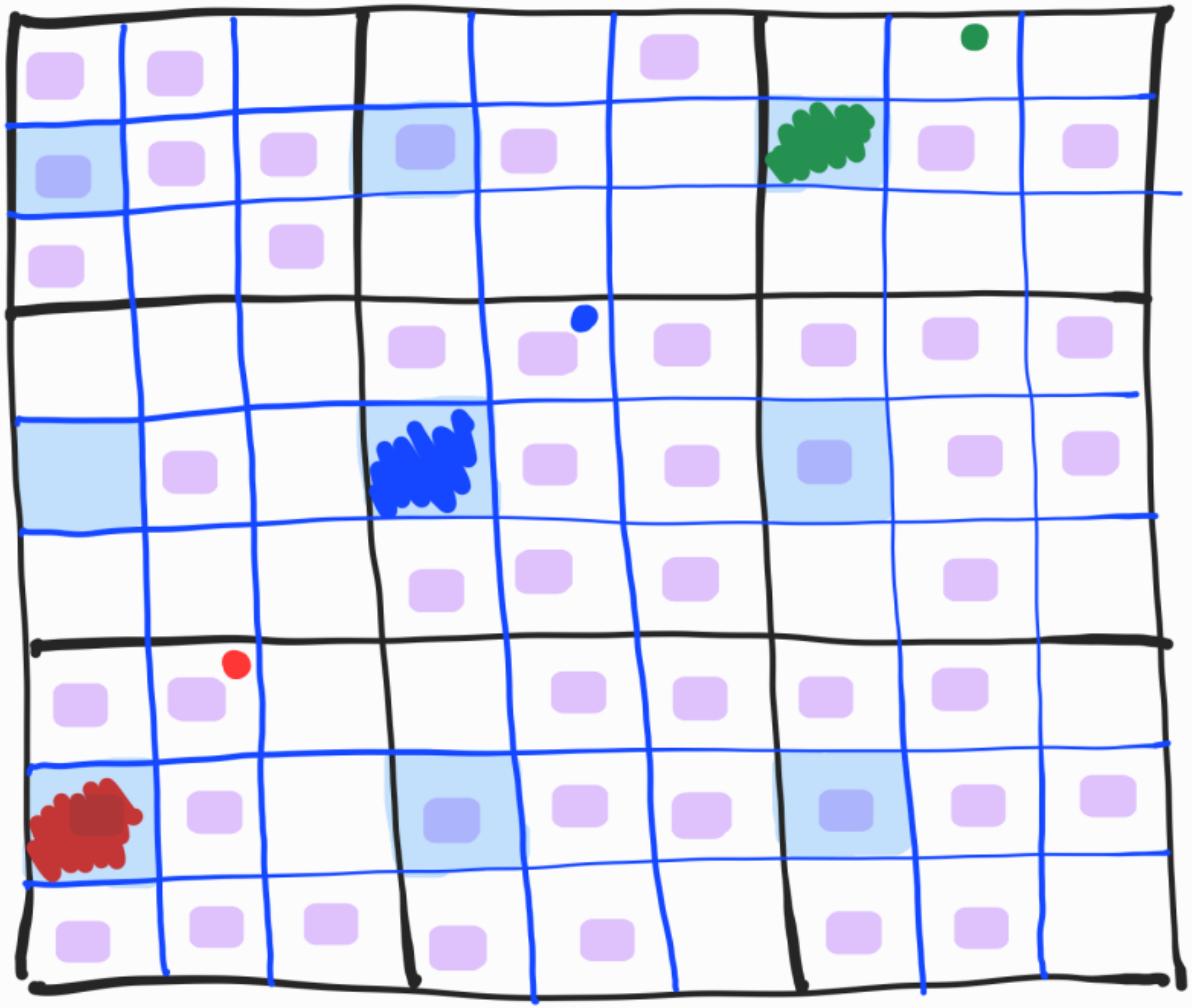
INDUCED ARITHMETIC REMOVAL

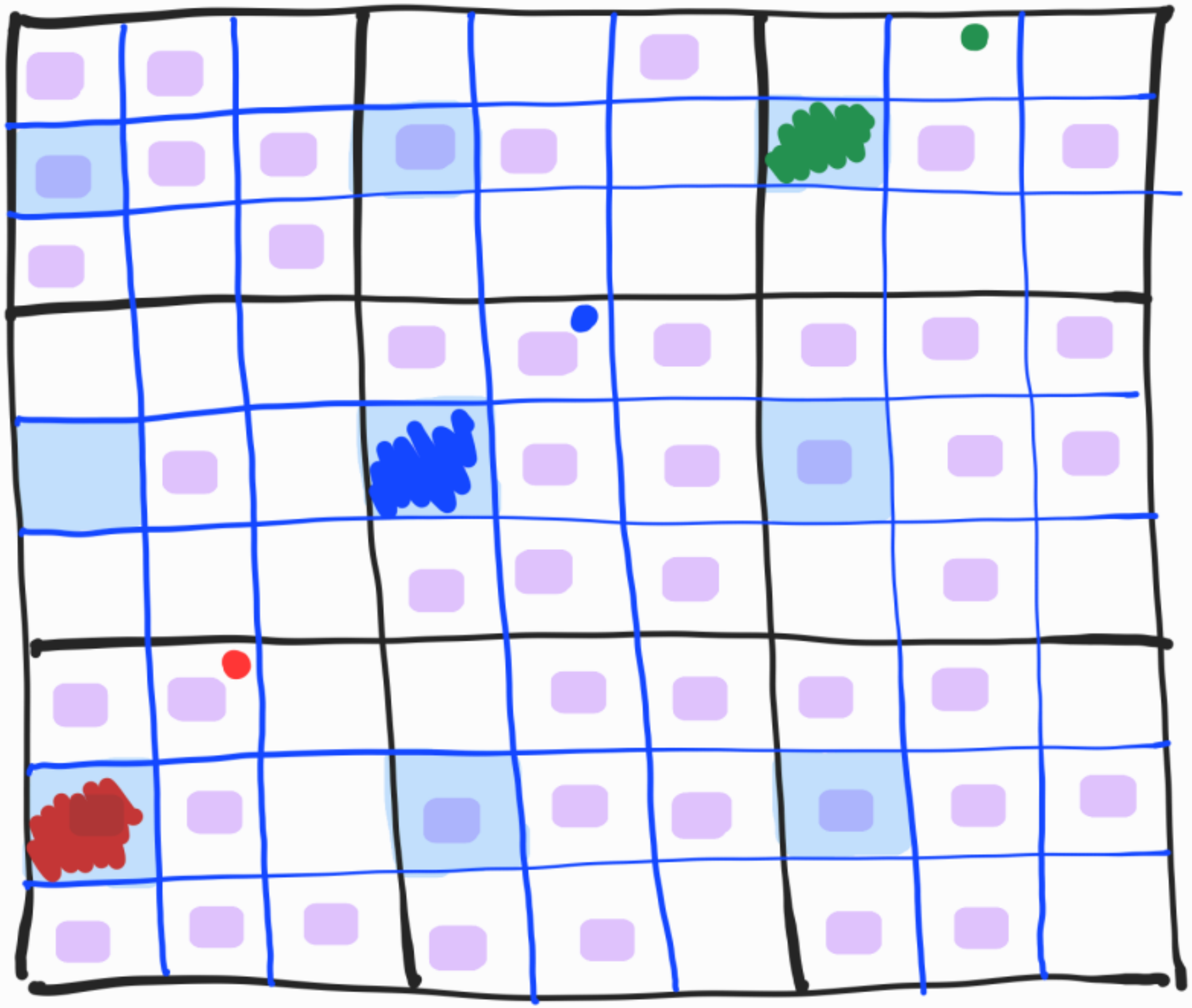


Suppose there is

x, $x+d$, $x+2d$

Pick subcosets to form an affine subspace

INDUCED ARITHMETIC REMOVAL



Suppose there is

x, $x+d$, $x+2d$

Pick subcosets to form an affine subspace

$$x,\ x+d,\ x+2d$$

$$\Downarrow$$

$$x+z,\ x+d+z,\ x+2d+z$$

is a 3-AP

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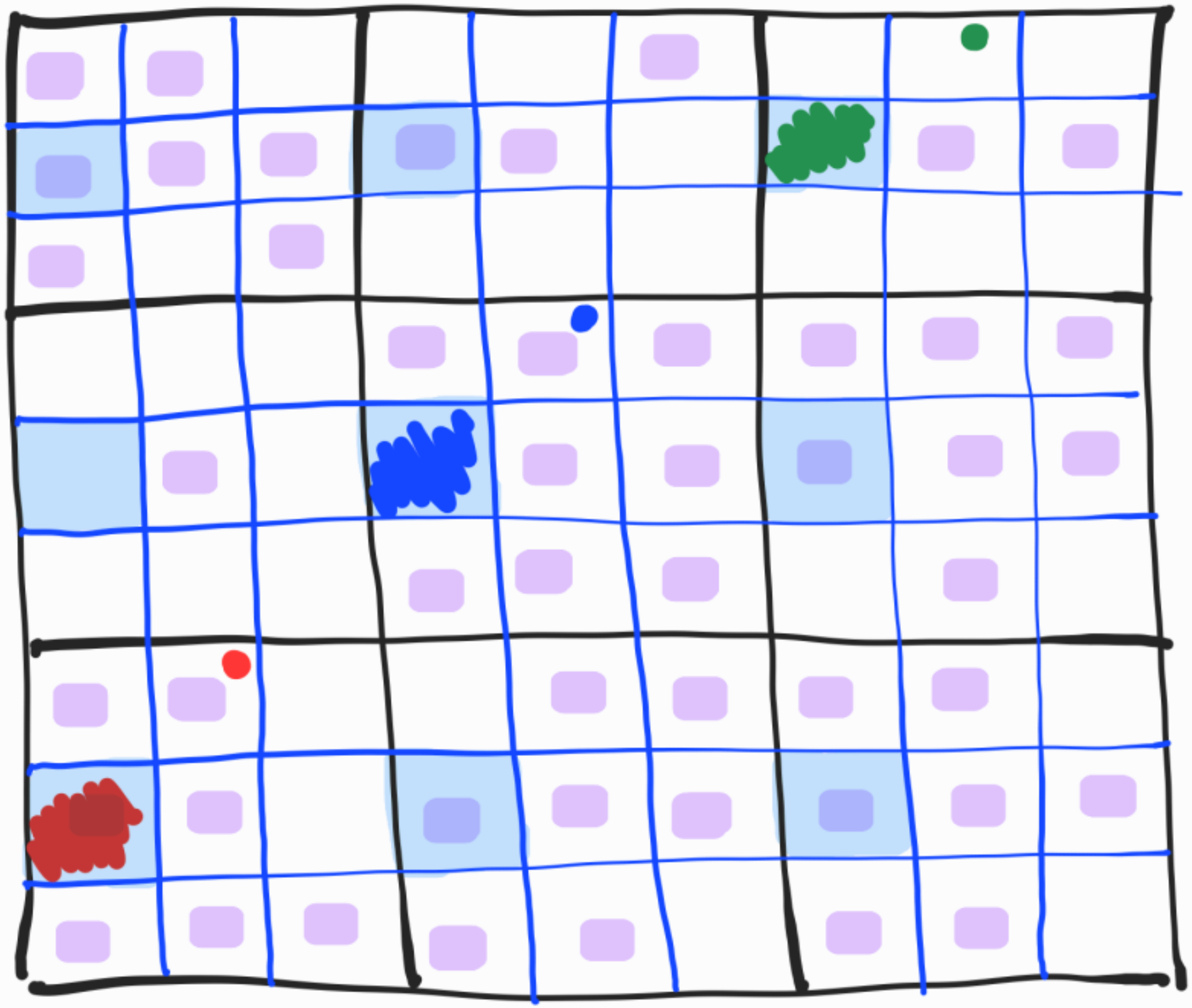

Suppose there is

x, $x+d$, $x+2d$

Pick subcosets to form an affine subspace

x, $x+d$, $x+2d$ ✓

$\Downarrow$

$x+z$, $x+d+z$, $x+2d+z$ is a 3-AP

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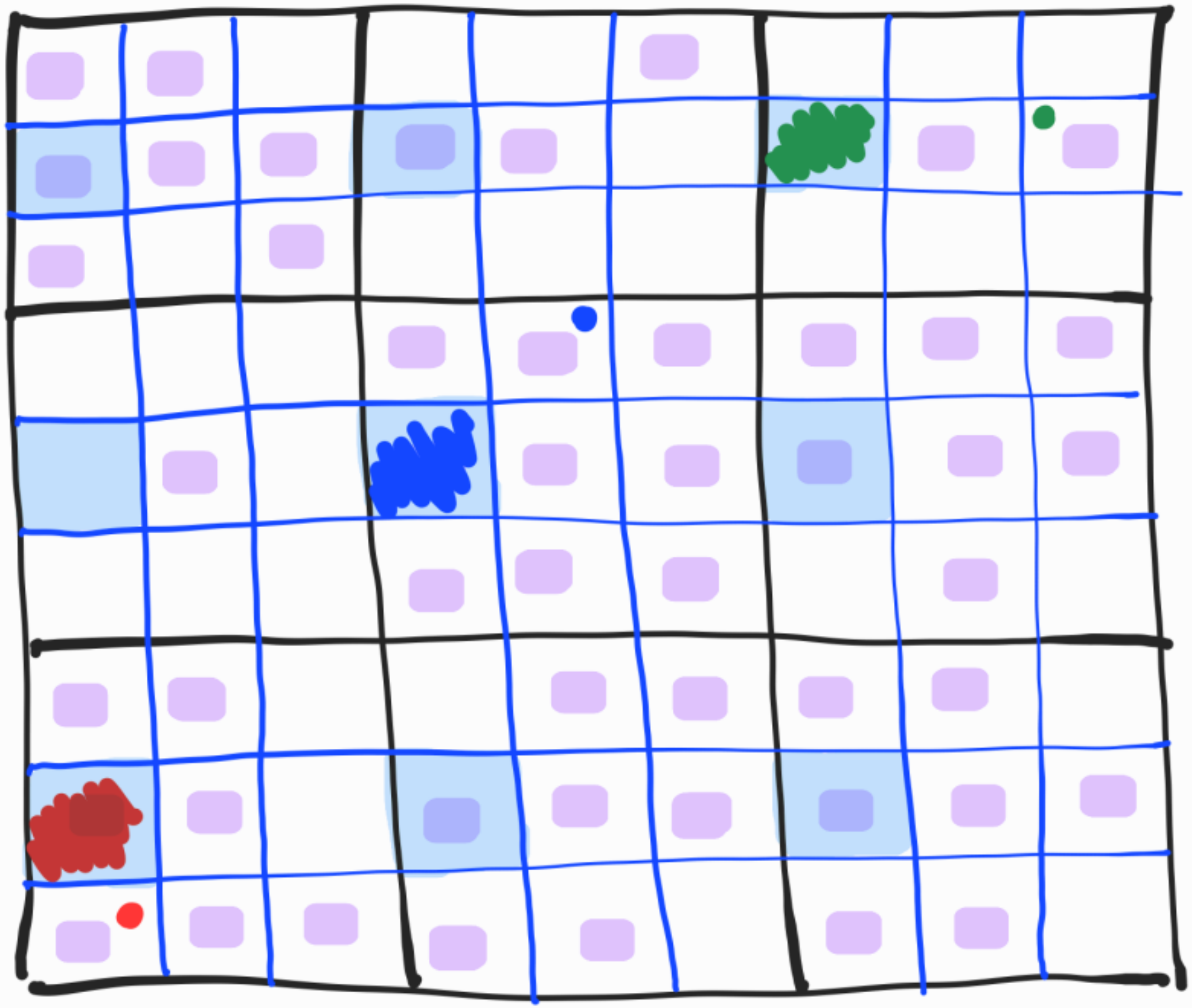
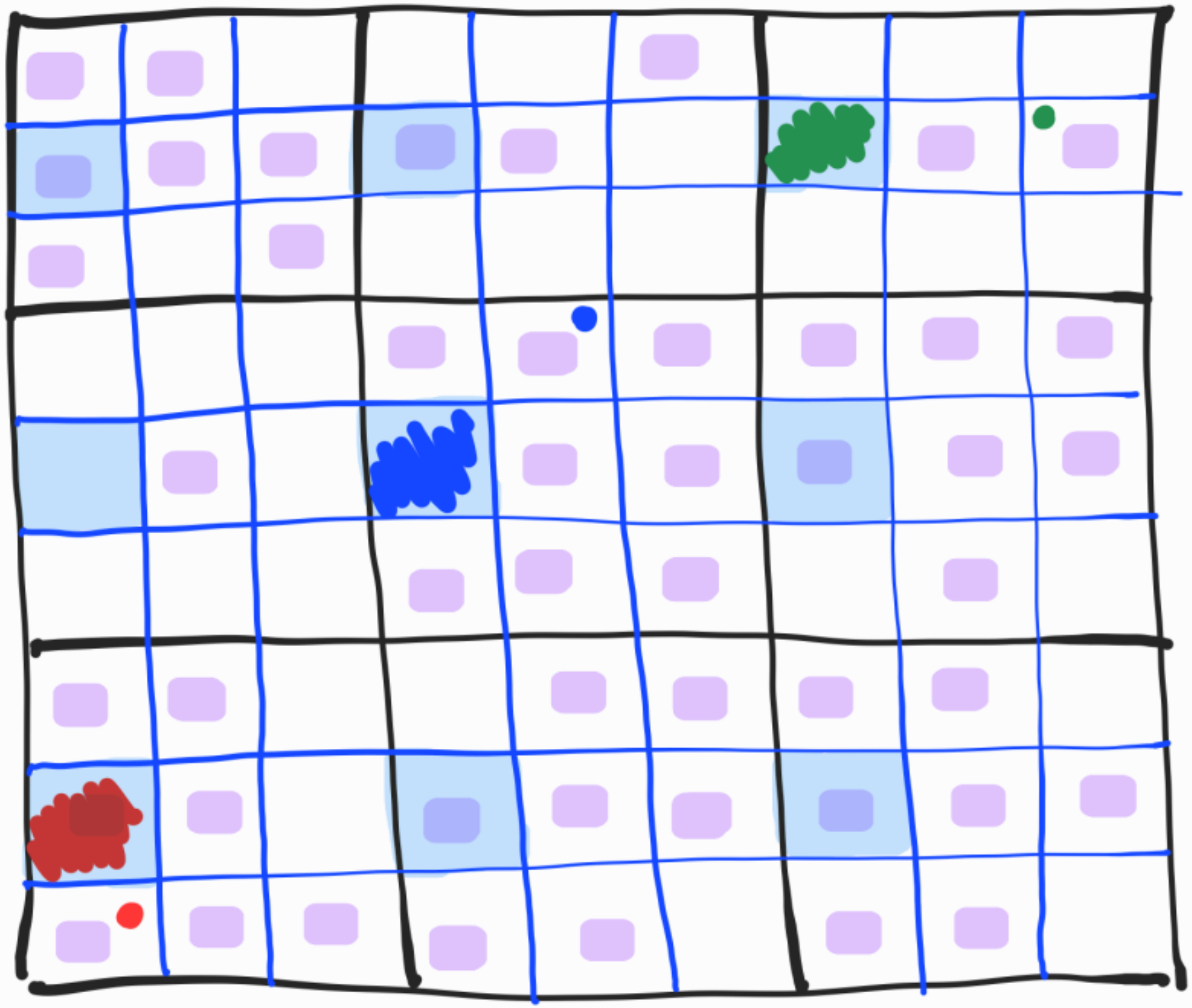


$$x + y + z = 0$$

INDUCED ARITHMETIC REMOVAL



$\underline{x} + \underline{y} + \underline{z} = 0$

(x, y, z) solution

$\not\Downarrow$

$(x+h, y+h, z+h)$ solution

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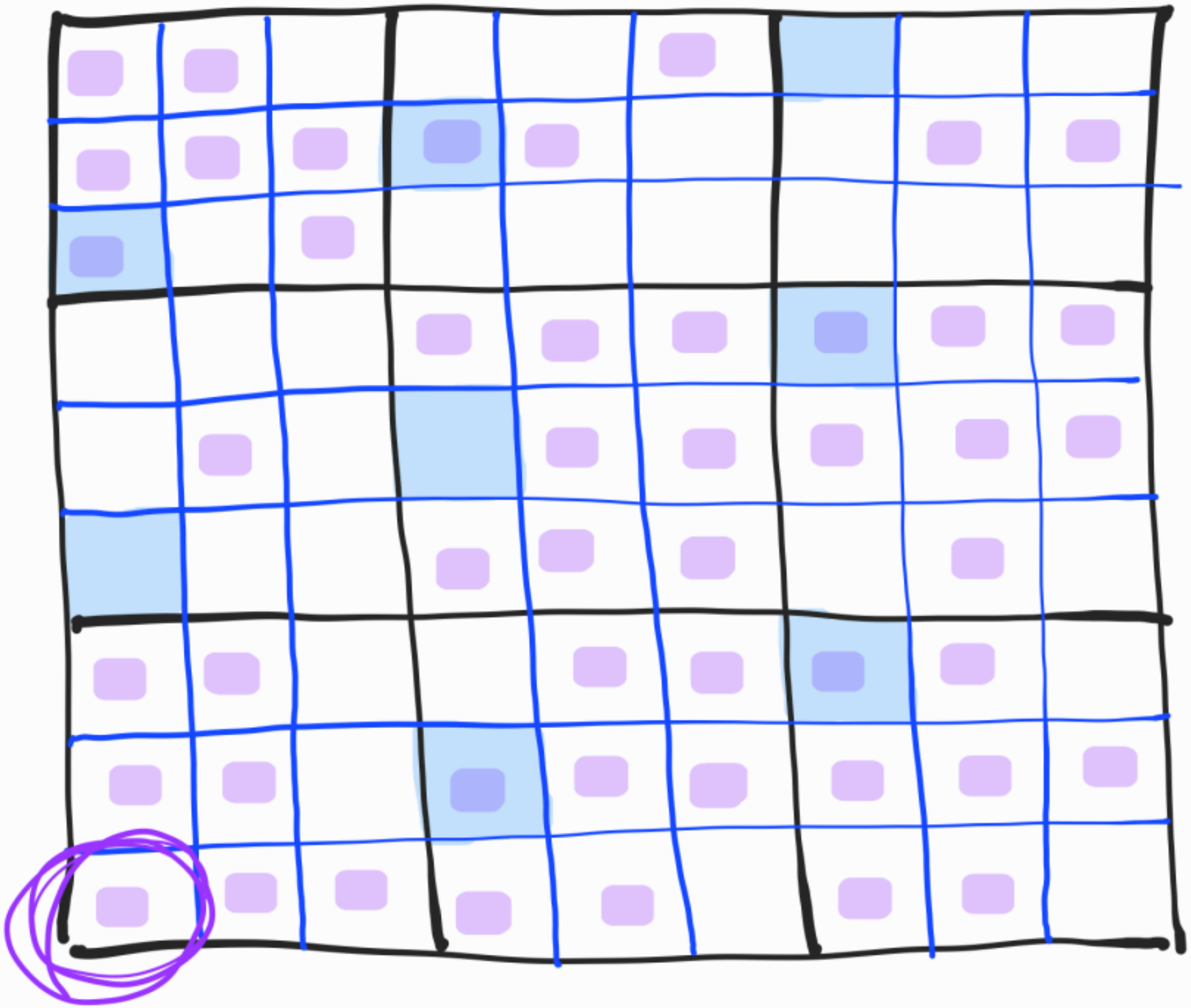
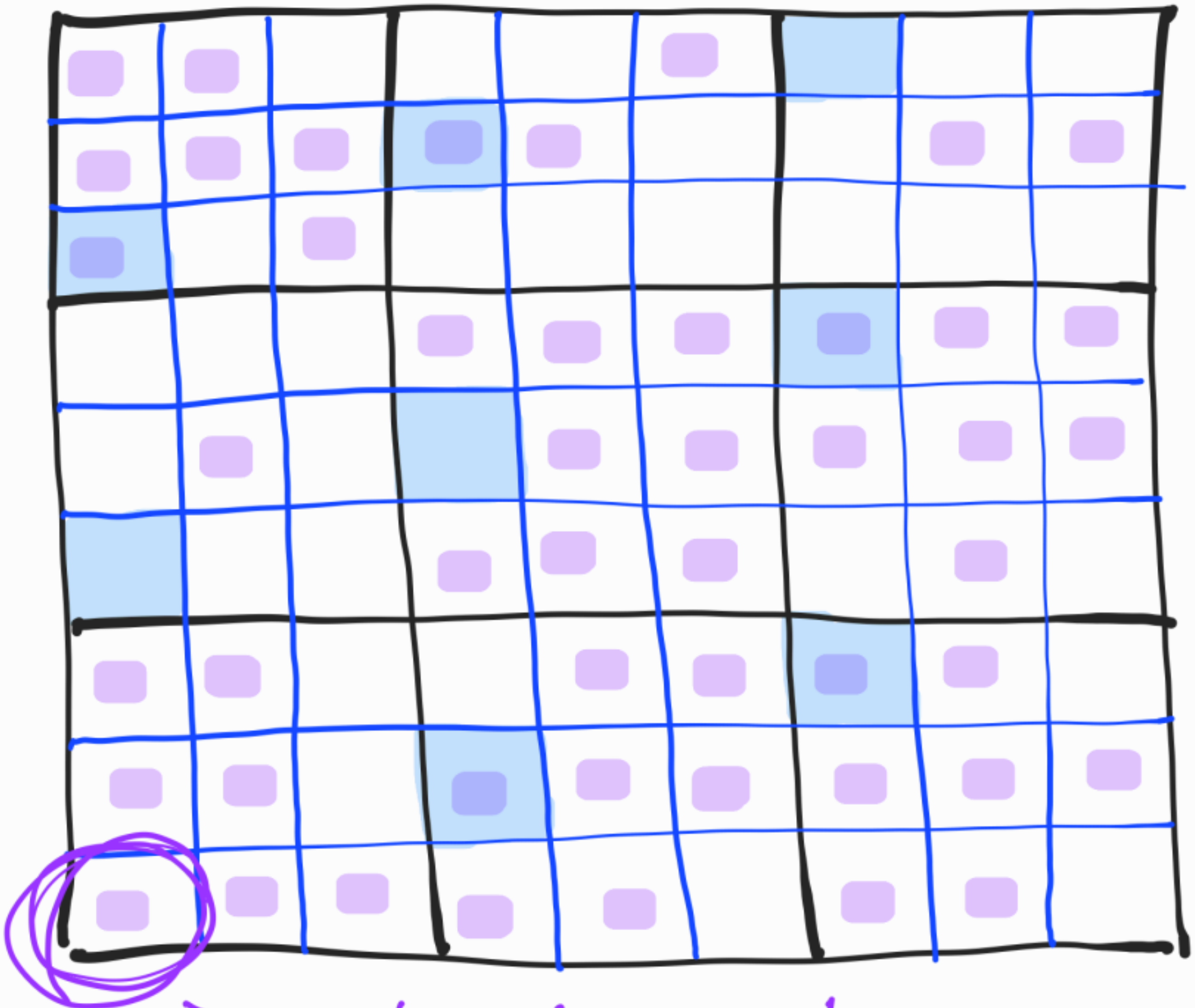


$$x + y + z = 0$$

Pick subcosets to form a linear subspace

INDUCED ARITHMETIC REMOVAL



$x + y + z = 0$

Pick subcosets to form a linear subspace

No choice!

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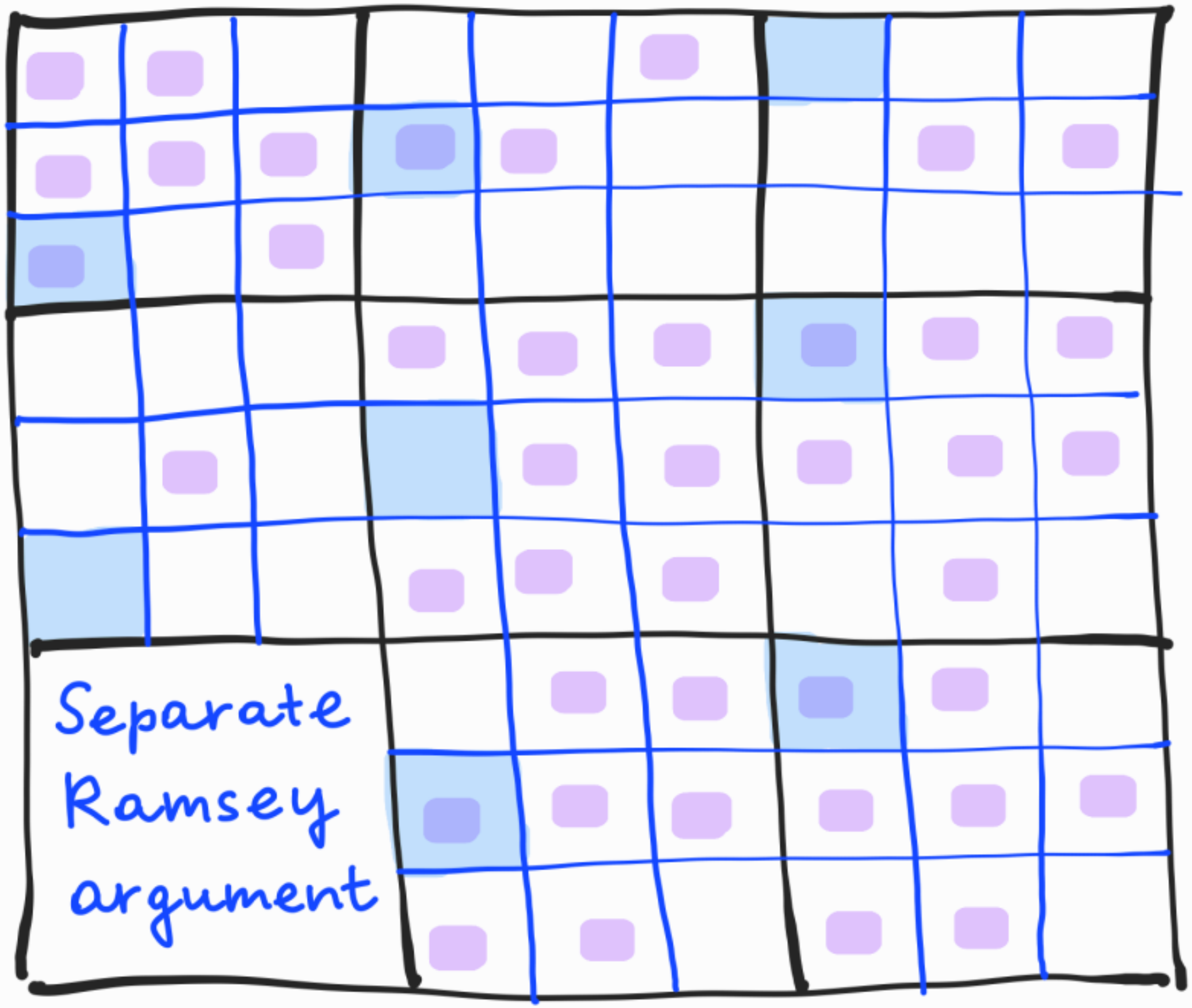

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Pick subcosets to form a linear subspace

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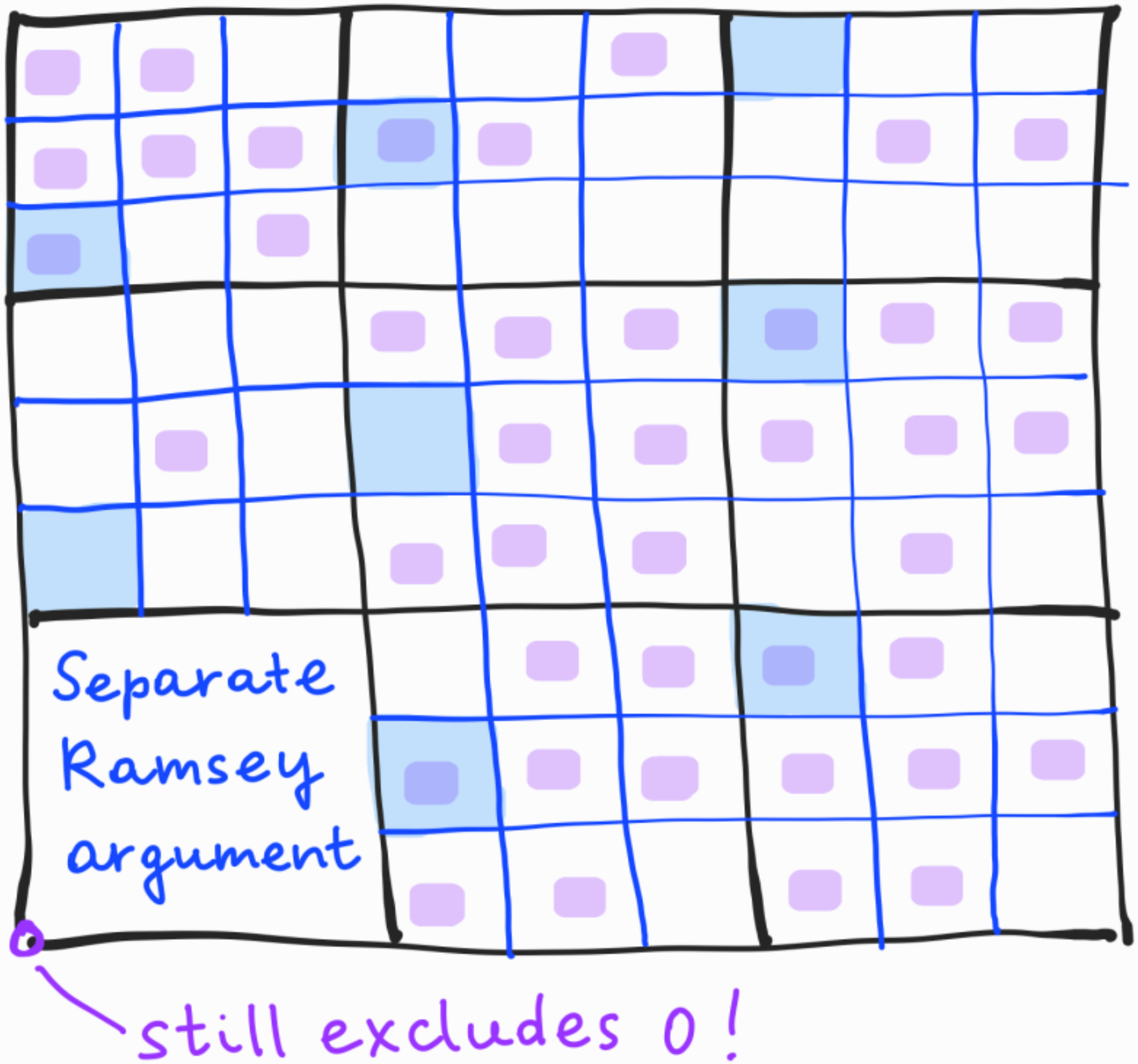
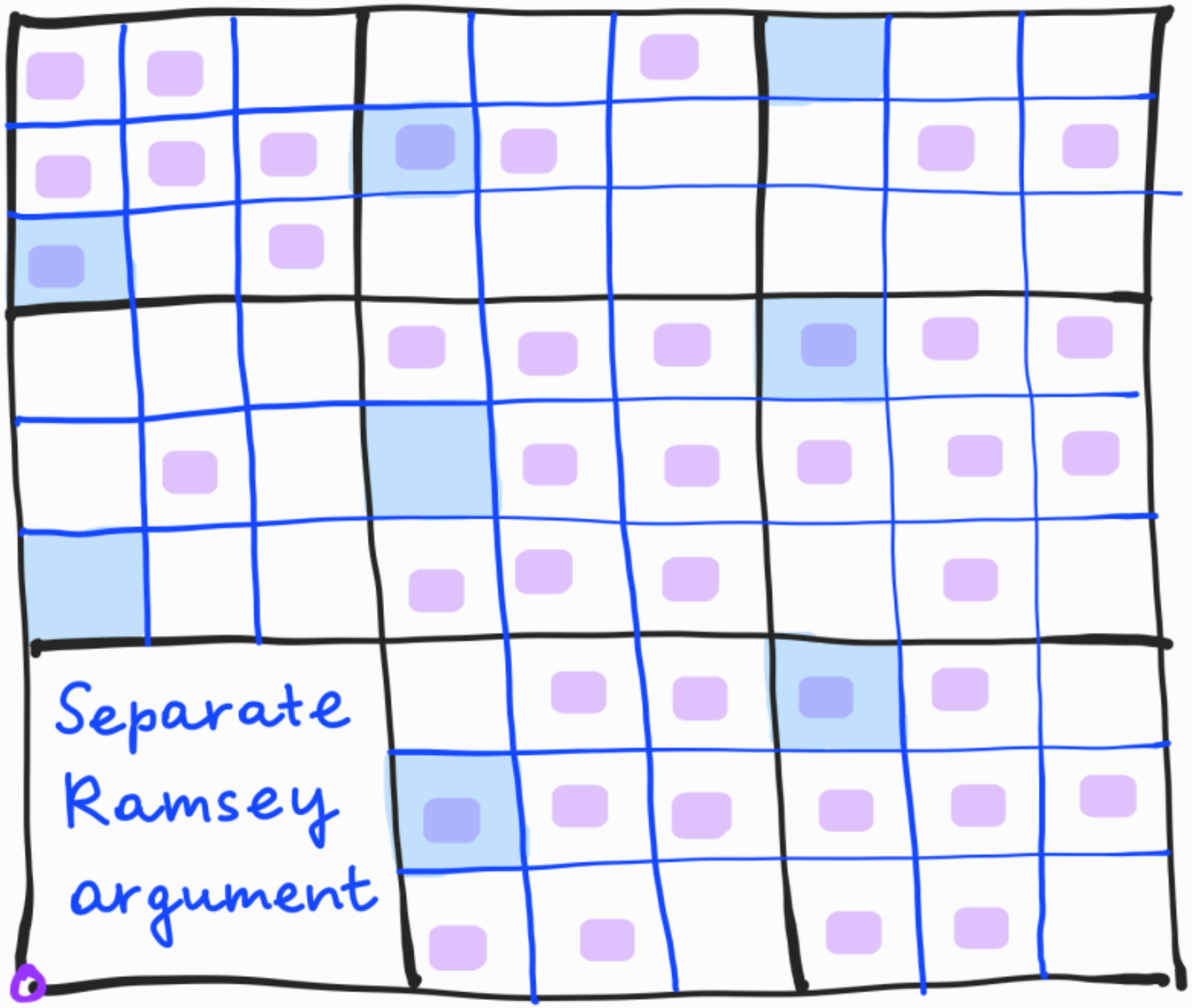


$x + y + z = 0$

Pick subcosets to form a linear subspace

INDUCED ARITHMETIC REMOVAL



still excludes 0!

$$x + y + z = 0$$

Pick subcosets to form a linear subspace

$\checkmark$ish $(\mathbb{F}_p^n \setminus \{0\})$

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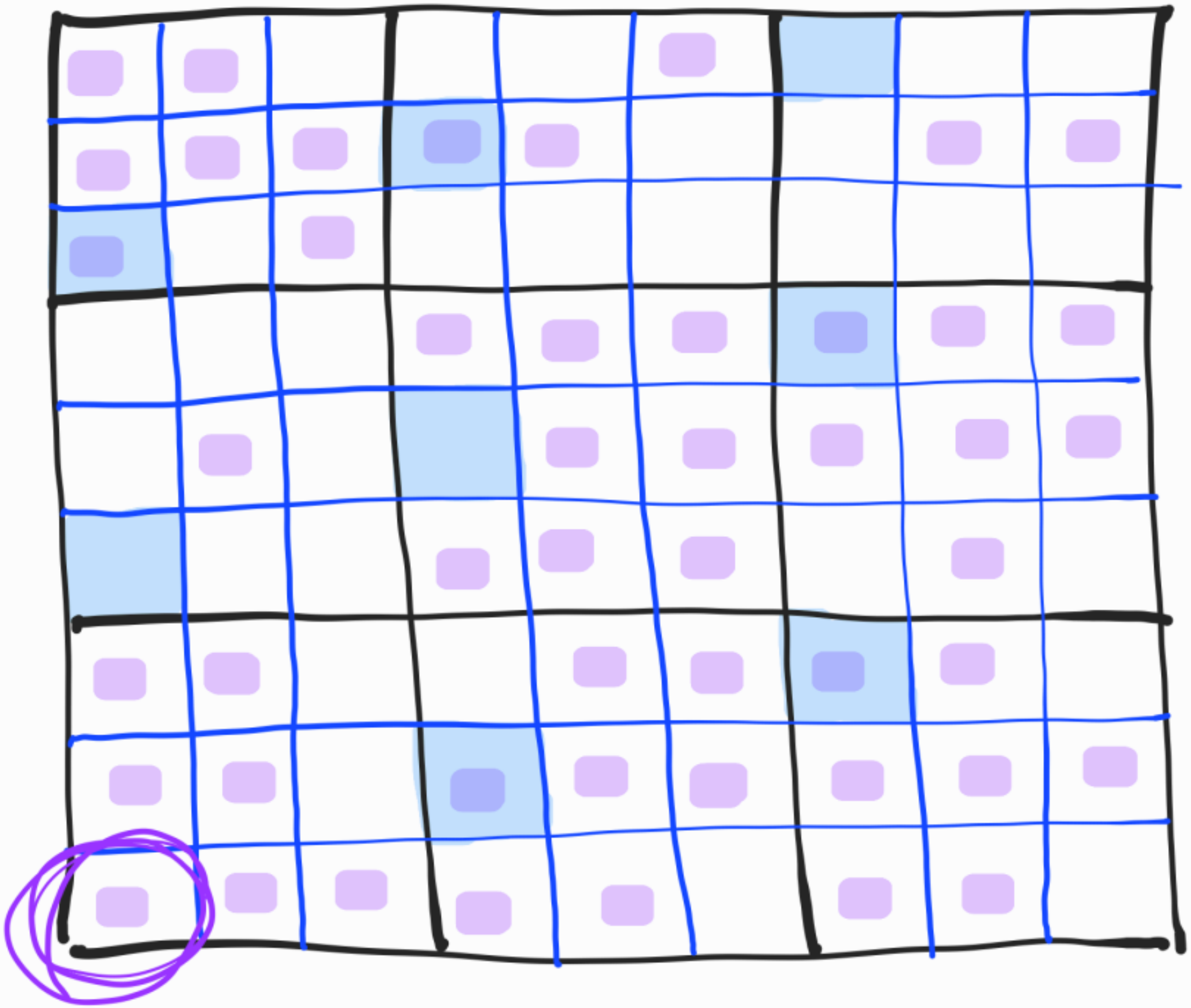
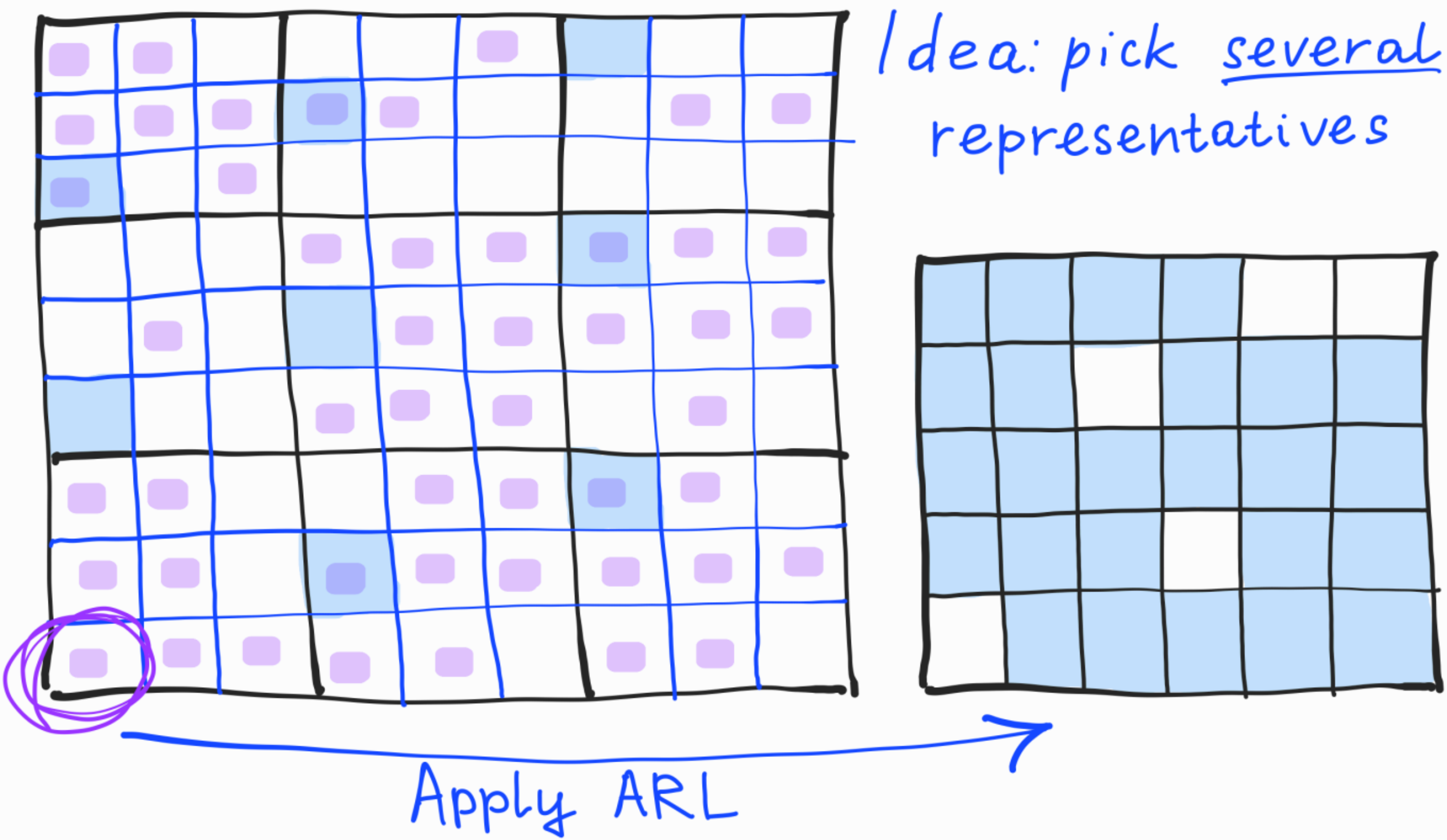


Idea: pick <u>several</u> representatives

INDUCED ARITHMETIC REMOVAL



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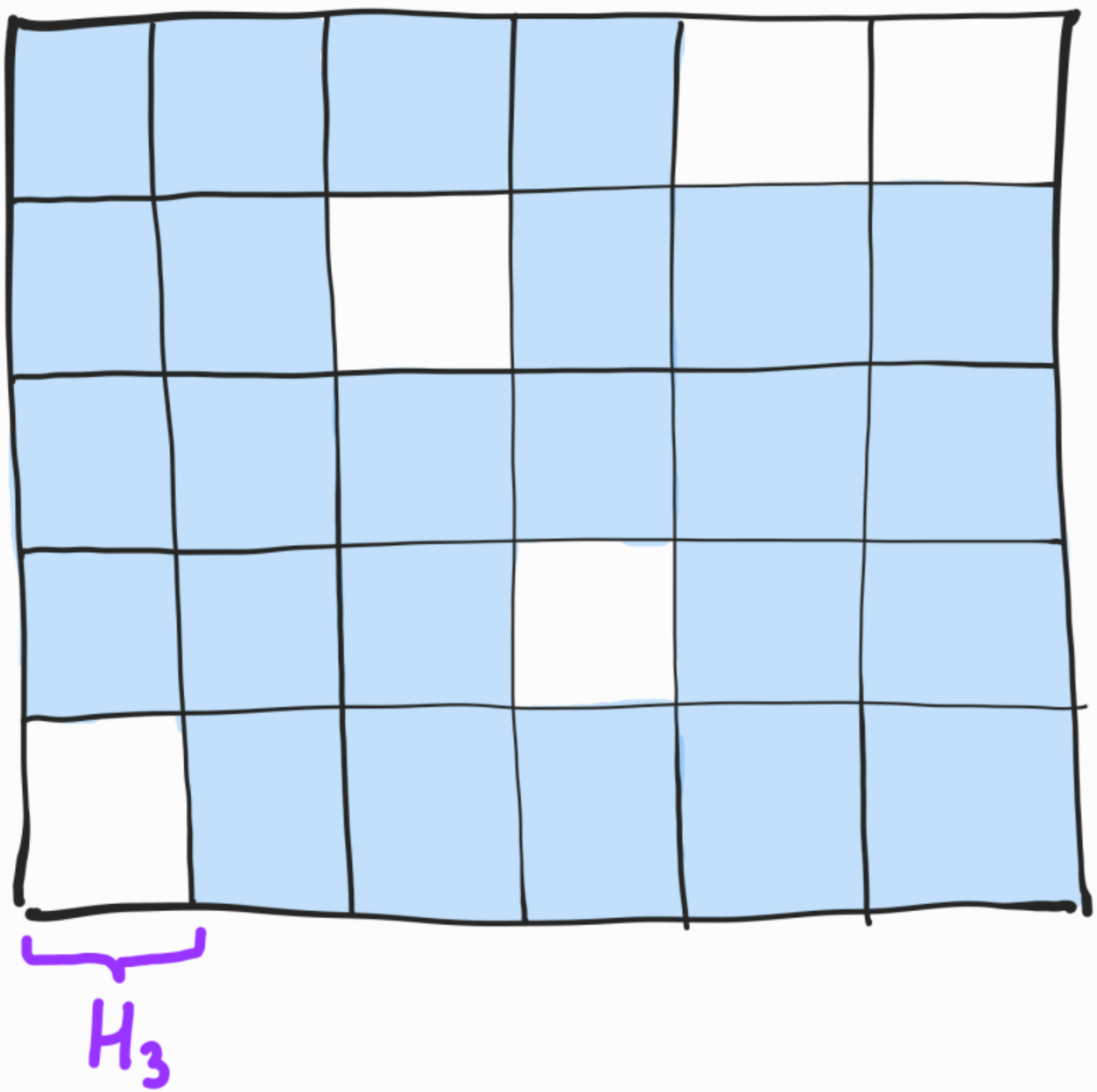

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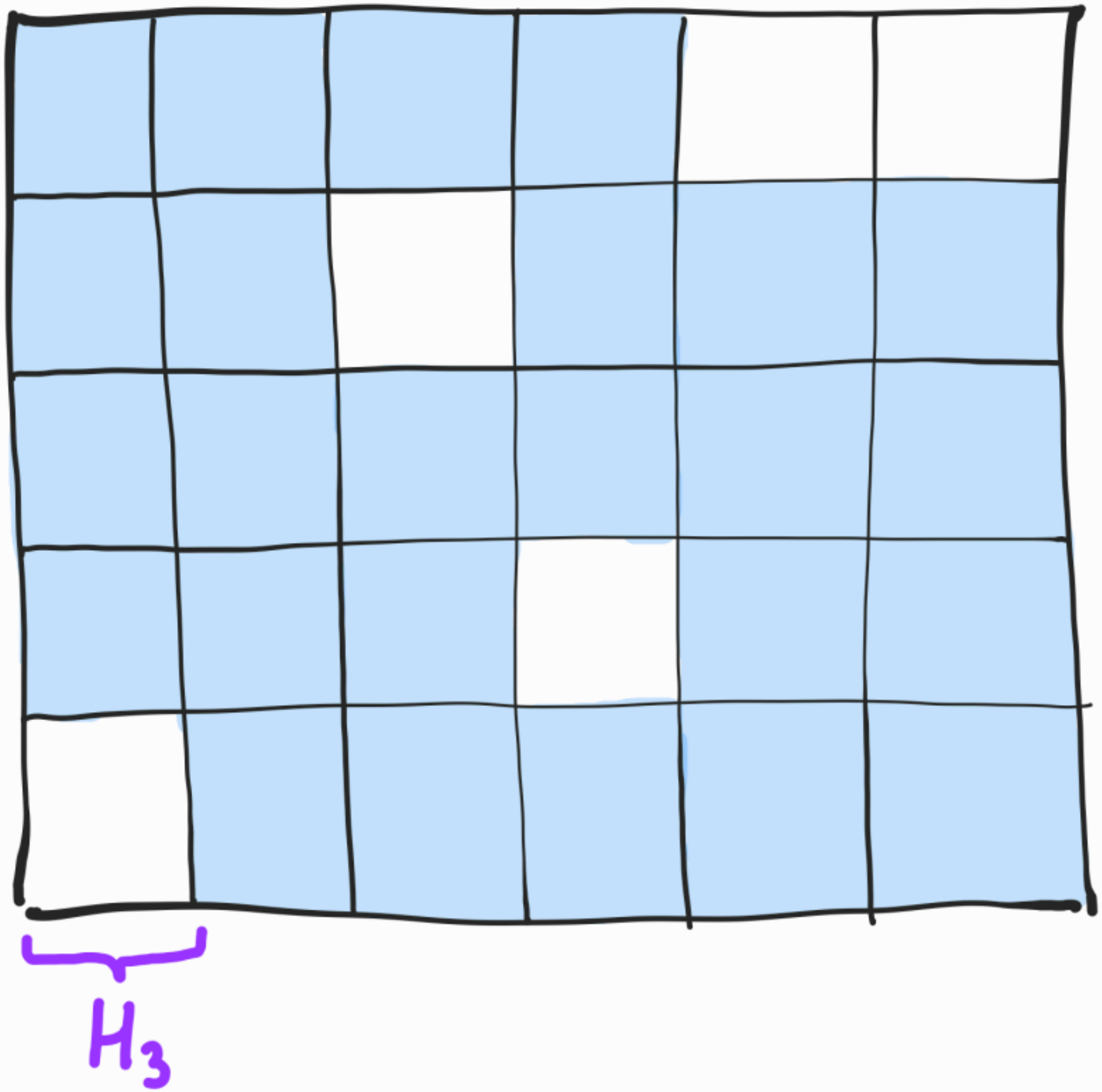

$$x + y - z = 0$$

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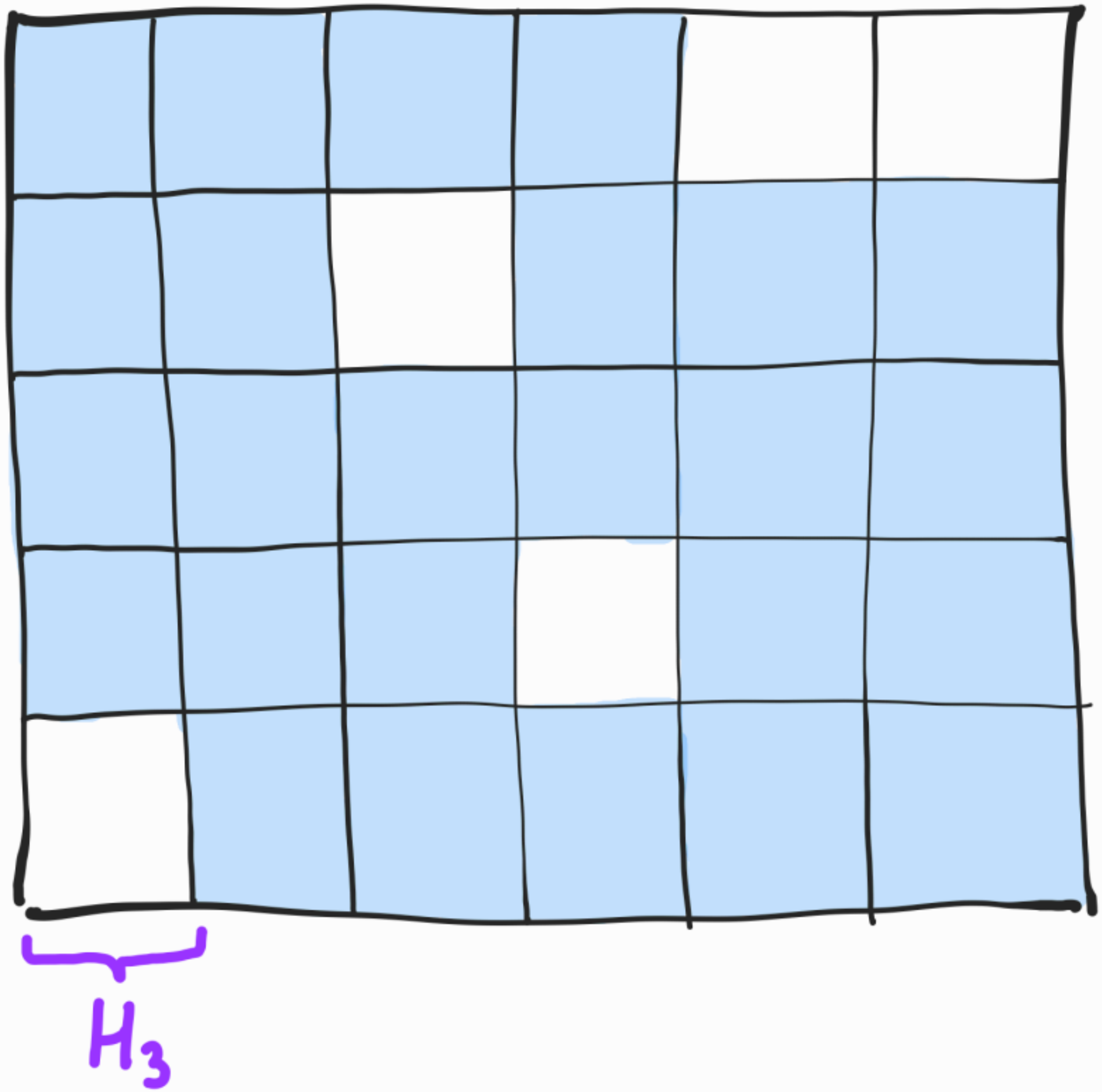
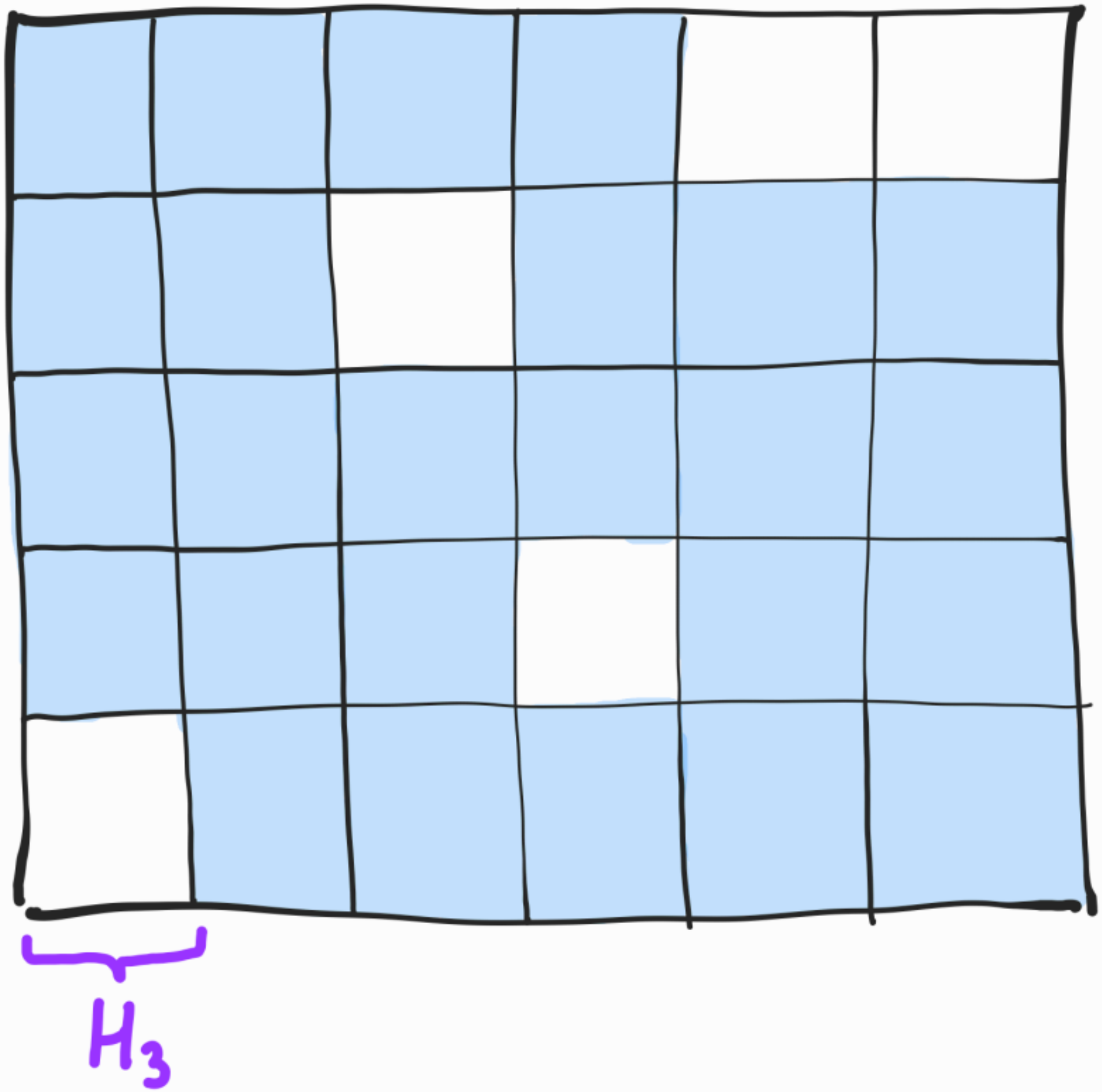


$$x + y - z = 0$$

3 representatives s.t.

- form solution to eq-n
- all regular
- agree with each other about densities

INDUCED ARITHMETIC REMOVAL



$x + y - z = 0$

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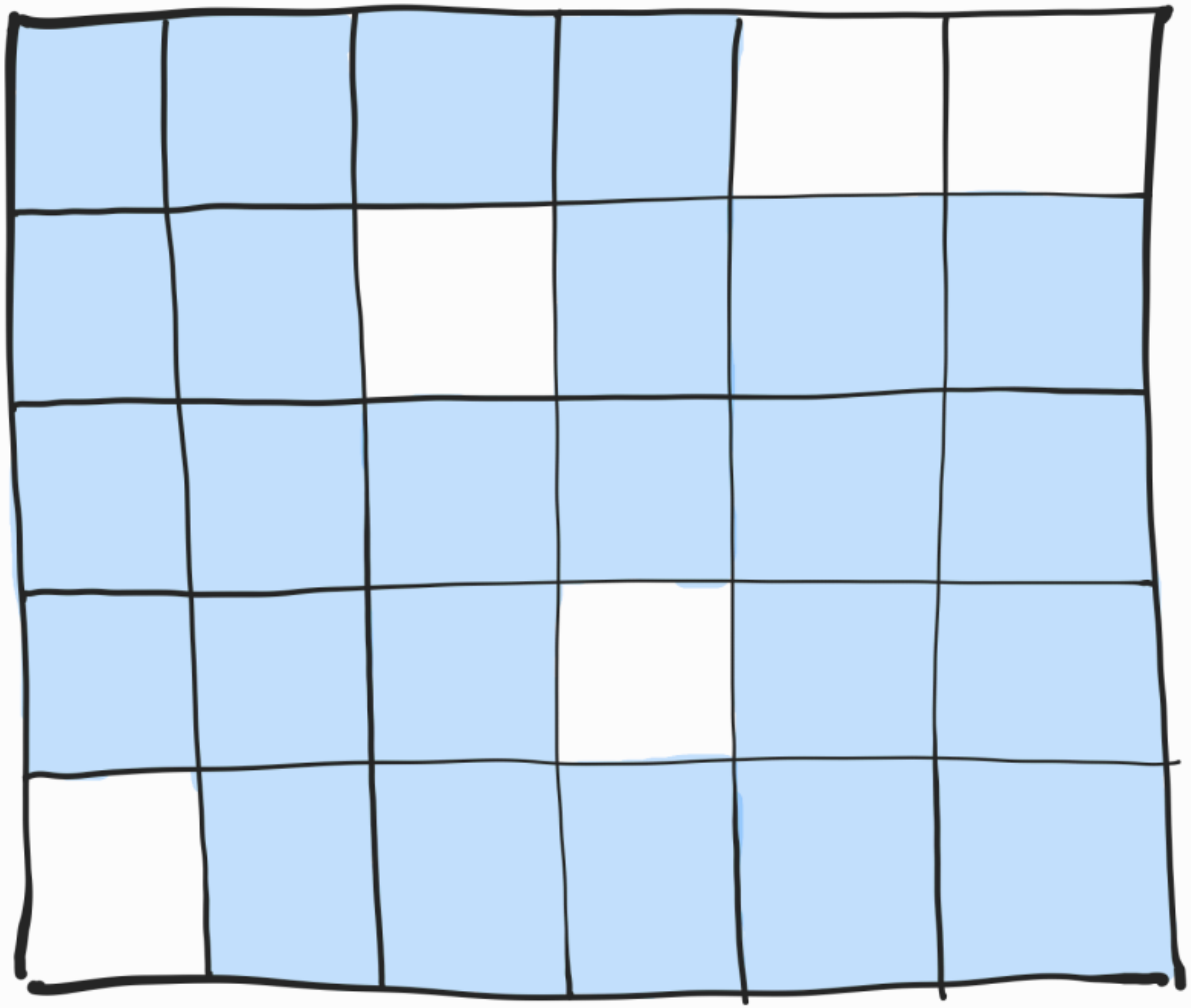
Ψ – auxiliary colouring

$$\Psi(H_3+c) = \left\{\begin{array}{l}\text{colours with high} \\ \text{density on } H_3+c\end{array}\right\}$$

INDUCED ARITHMETIC REMOVAL



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Serra, Vena 2014: positive proportion of solutions is monochromatic

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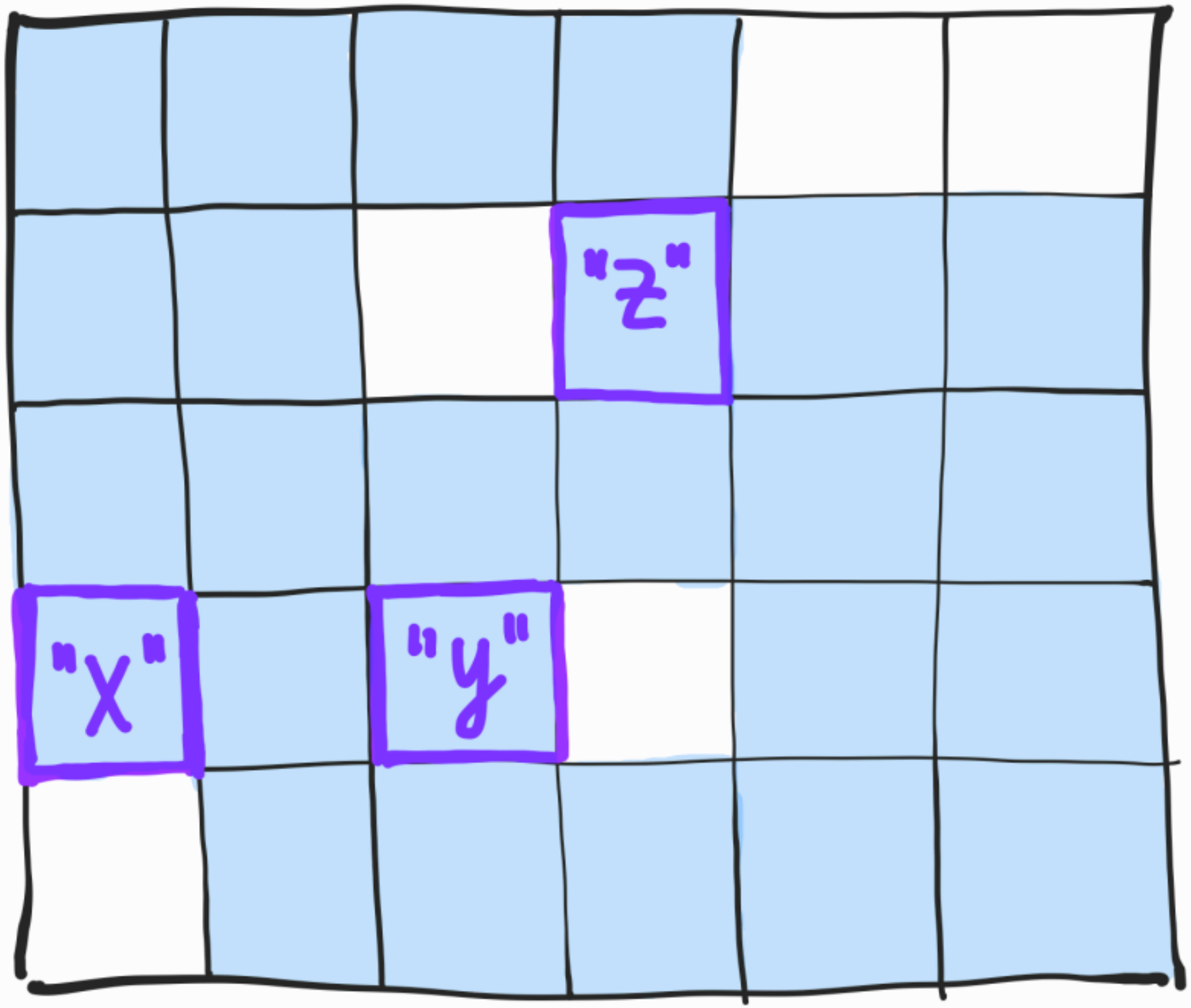


$x + y - z = 0$

3 representatives s.t.

- form solution to eq-n
- all regular
- agree with each other about densities

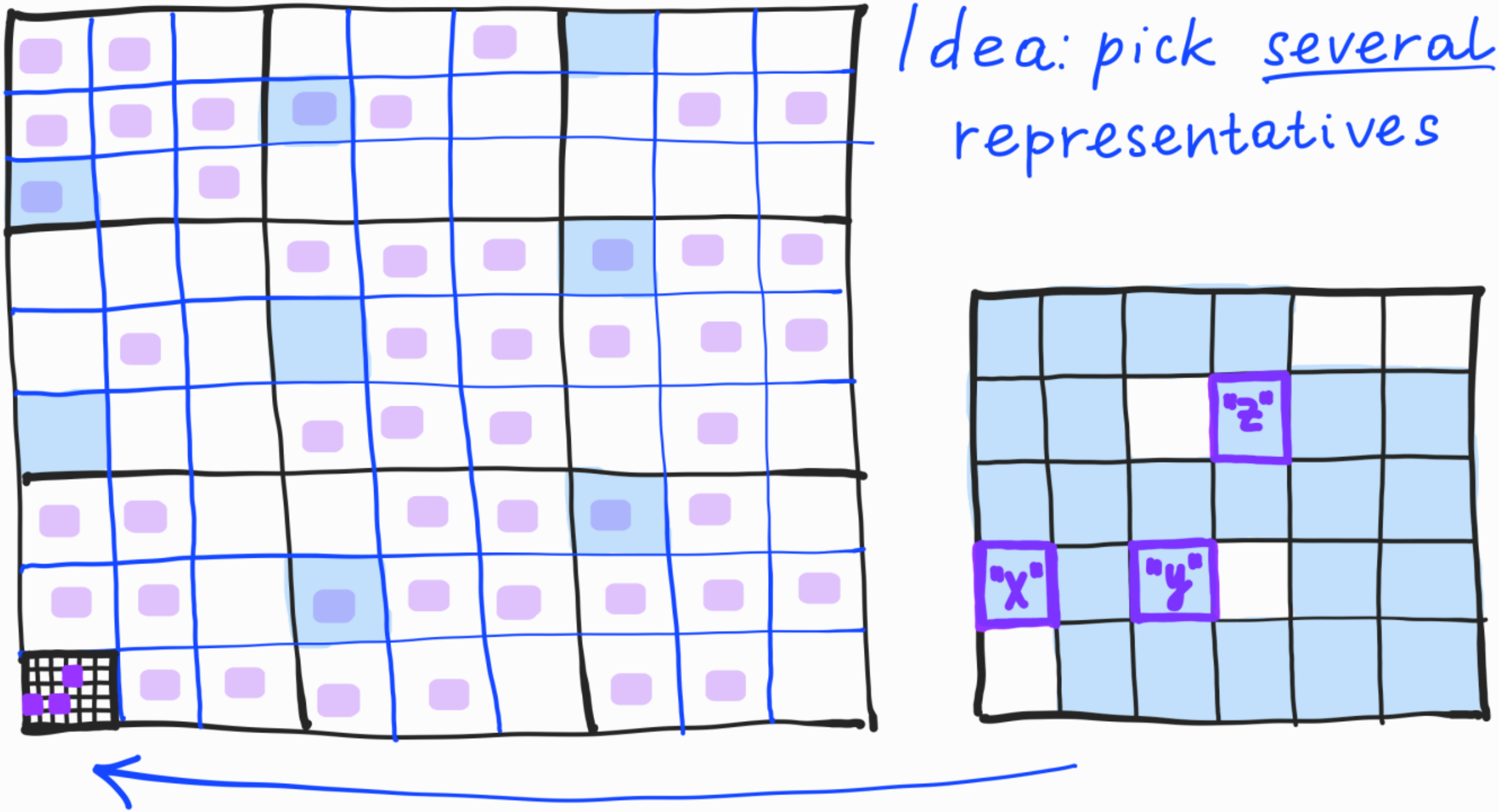
ψ - auxiliary colouring

$\psi(H_3 + c) = \left\{ \begin{matrix} \text{colours with high} \\ \text{density on } H_3 + c \end{matrix} \right\}$

Serra, Vena 2014: positive proportion of solutions is monochromatic

INDUCED ARITHMETIC REMOVAL

Idea: pick <u>several</u> representatives



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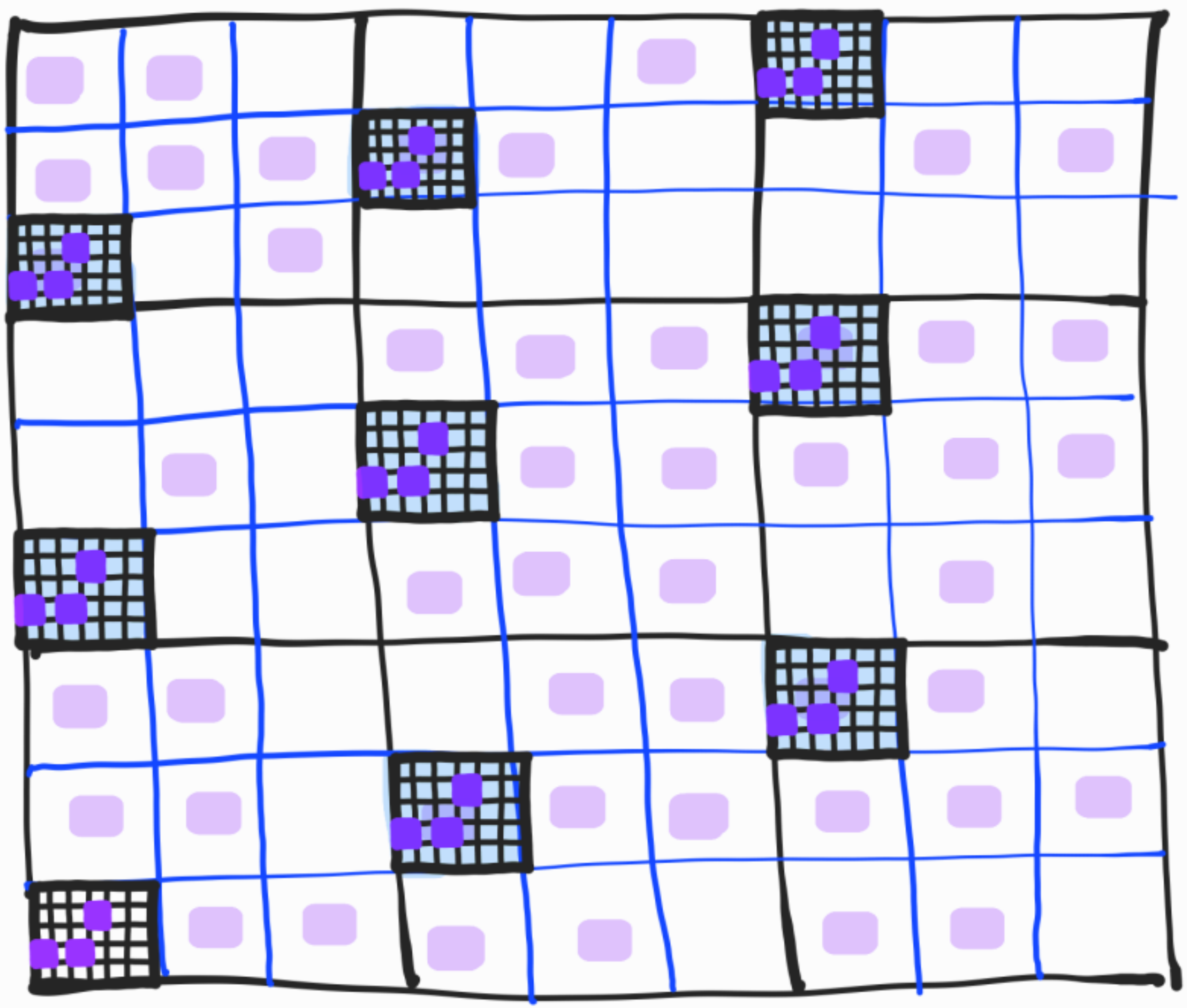

"Same" representatives in other cosets of H,

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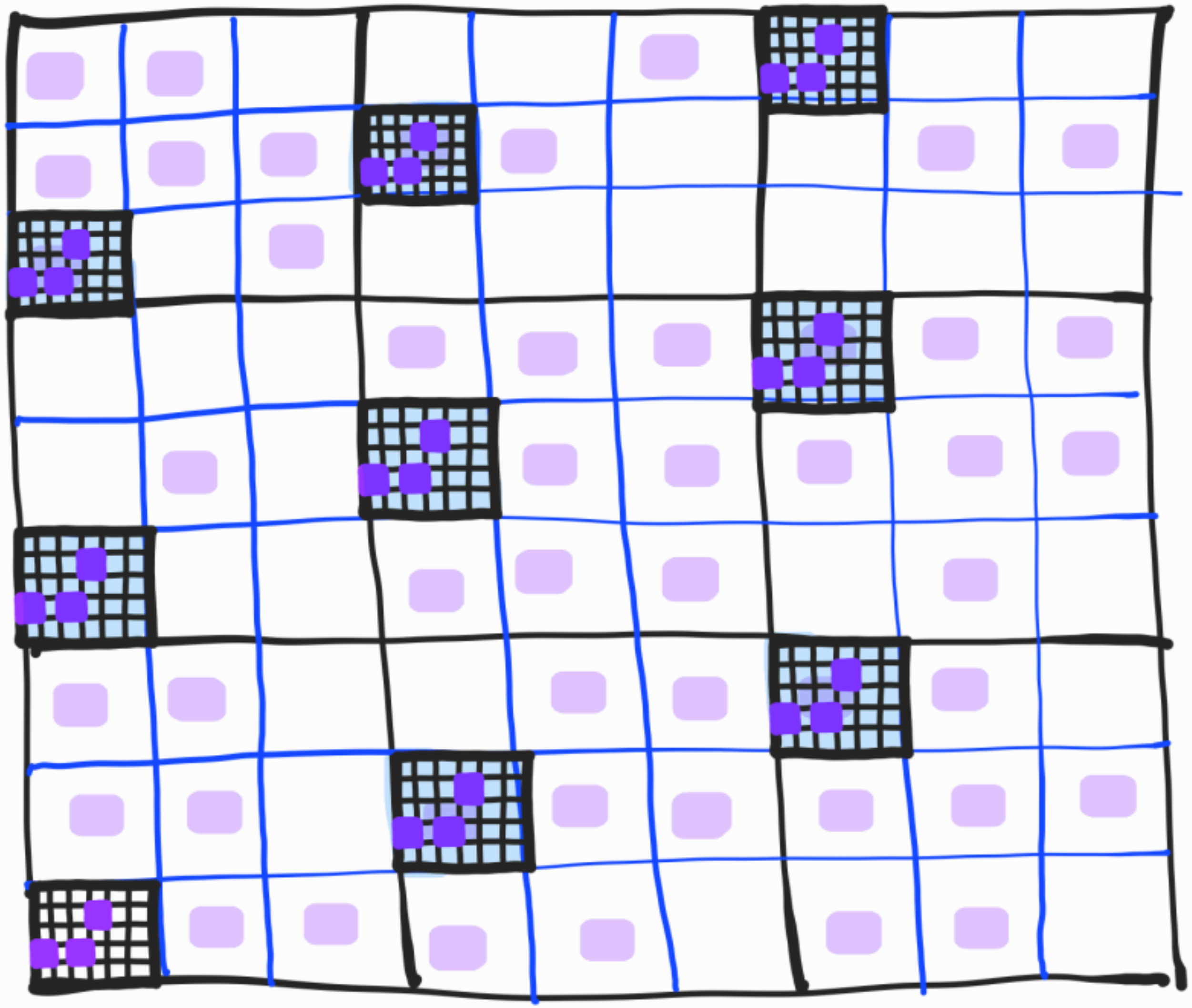
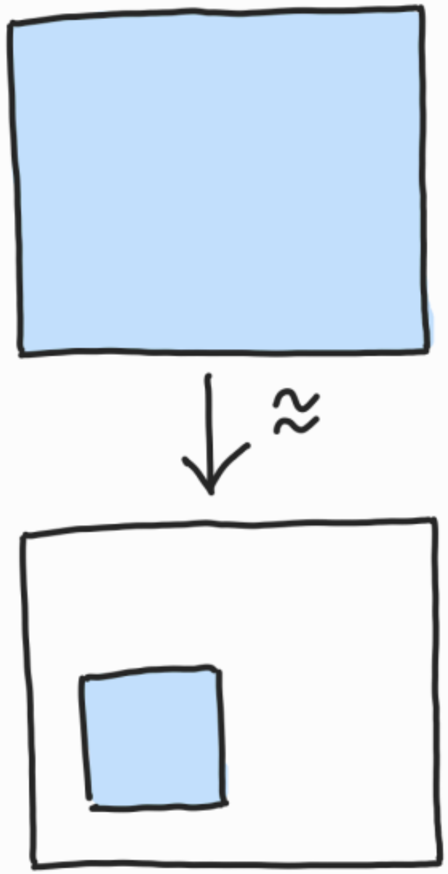


"Same" representatives in other cosets of H,



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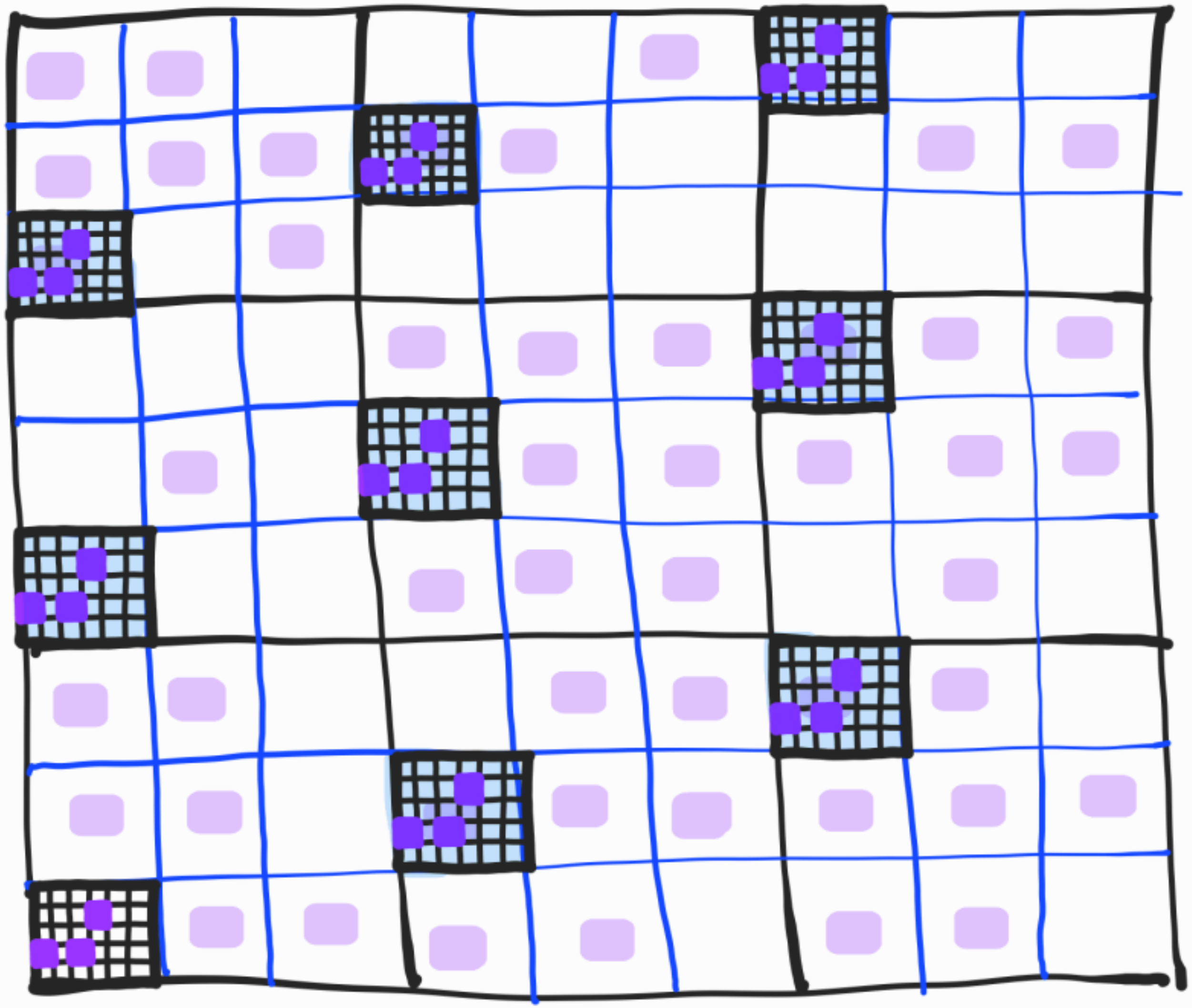

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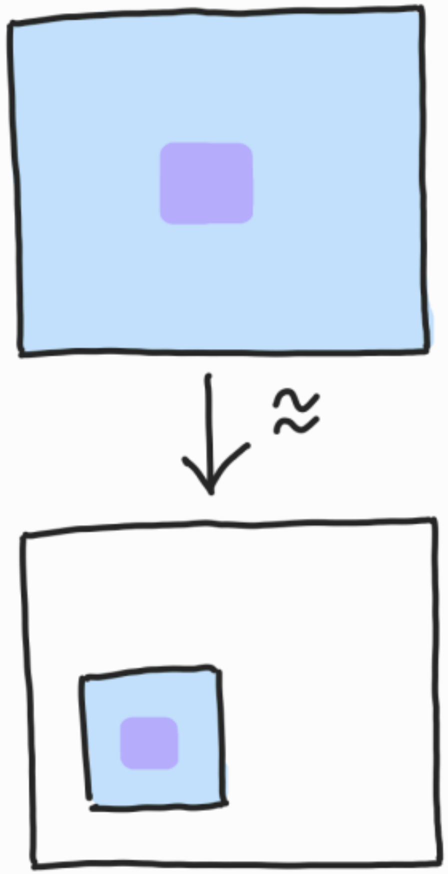

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Conjecture

Let $\mathcal{H}$ be a partition-regular arithmetic pattern of any true complexity. For all $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $\chi : \mathbb{F}_p^n \to [r]$ satisfies $\Lambda_{\mathcal{H}}(\chi) < \delta$, then χ can be made $\mathcal{H}$-free by recolouring $\leq \varepsilon |\mathbb{F}_p^n|$ elements.

Thank you for your attention! :)

email: vg338@cam.ac.uk

arxiv: 2412.15170